A CHARACTERIZATION OF THE RADICAL OF AN ALGEBRA

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1. The first main result.

We shall prove the following result.

THEOREM 1. *Let F be any field and A an algebra over F with a unity element. Then the radical of A consists of all elements h such that $g+h$ is regular for every regular g.*

Let H be the set of all elements h defined in the theorem. It is easy to see that H is a linear set over F. We shall prove now that if A is simple, $H=0$.

Let g and g_1 be any regular elements of A and h be in H. Then $g_1^{-1}g+h$ is regular so that $g+g_1h$ is regular. Hence g_1h is in H and similarly hg_1 is in H. An arbitrary element a of A has[1] the form $a=\sum_{i=1}^{n} g_i$ with regular elements g_i so that $ah=\sum g_ih$ is a sum of elements g_ih of H. Thus ah, and similarly ha, is in H so that H is an ideal of A. If $H\neq 0$ then $H=A$ since A is simple. But A contains the regular element -1, and $(-1)+1$ is not regular so that 1 cannot be in H, whence $H\neq A$. Hence $H=0$.

Next we shall prove that $H=0$ whenever A is semi-simple. Now $A=A_1+A_2+\cdots+A_t$ where the A_i are simple, and each x of A has a unique expression $x=a_1+a_2+\cdots+a_t$ with a_i in A_i. Further, x is regular if and only if each a_i is a regular element of A_i. Let $g=g_1+\cdots+g_t$ be regular, $h=h_1+\cdots+h_t$ be in H, so that $g+h=(g_1+h_1)+\cdots+(g_t+h_t)$. Then $g+h$ is regular for every regular g if and only if g_i+h_i is regular in A_i for every regular g_i of A_i. By the proof above for simple algebras every $h_i=0$ so that $h=0$ and $H=0$.

In considering the case of a general algebra A, we show first that the radical R is contained in H. Let g be regular and r lie in R. Then $g+r$ is regular if and only if $1+g^{-1}r$ is regular. Now $g^{-1}r$ is in R, $(g^{-1}r)^t=0$ for some integer t, $(g^{-1}r)^{2t+1}+1=1$. If λ is an indeterminate, $\lambda+1$ is a factor of $\lambda^{2t+1}+1$ so that $g^{-1}r+1$ is a factor of $(g^{-1}r)^{2t+1}+1=1$; hence, $g^{-1}r+1$ is regular, $g+r$ is regular, r is in H, and R is contained in H.

It remains to prove that R contains H. Since $A-R$ is semi-simple, the set H_0 defined for $A-R$, similarly to H for A, is the zero set. If g is regular in A and h is in H, the class $[g+h]$ in $A-R$ is a regular ele-

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[1] K. Shoda, Mathematische Annalen, vol. 107 (1933), pp. 252–258.

ment of $A-R$. But $[g+h]=[g]+[h]$, and $[g]$ varies over all[2] regular elements of $A-R$ so that $[h]$ must be in $H_0=[0]=R$. Hence h is in R, H is contained in R, $H=R$, and the proof of the theorem is complete.

2. **Extension to arbitrary algebras.** The theorem above is applicable to algebras A_0 without a unity element in the sense that by adjoining a unity element to A_0 we do not alter its radical. To see this, let R_0 be the radical of A_0 and R the radical of the corresponding algebra A with a unity element. Every element of A has the unique form $a=\alpha+a_0$ with α in F, a_0 in A_0. If r_0 is in R_0 then $ar_0=\alpha r_0+a_0r_0$ is a sum of elements of R_0, ar_0 is nilpotent, r_0 is properly nilpotent in A, so that $R_0\leqq R$. Conversely, let r be in R so that $r^t=0$ for some integer t, $r=\sigma+s_0$ with σ in F, s_0 in A_0, $r^t=\sigma^t+s_1=0$ with s_1 in A_0. Then $s_1=0$, $\sigma=0$ so that $r=s_0$ in A_0. But r is properly nilpotent in A, hence in A_0, hence r is in R_0, $R\leqq R_0$, $R=R_0$.

While the fact just proved enables one to apply Theorem 1 to arbitrary algebras, nevertheless it is desirable to obtain a criterion not dependent on the unity element, as Professor Marshall Hall has pointed out to the author. The remainder of this section is devoted to this purpose.

If A is an algebra without a unity element, the symbol A' will be used throughout the paper to denote the algebra obtained from A by adjoining a unity element. If A has a unity element, A' is defined to be A.

DEFINITION. *An element x of an algebra will be called "quasi-regular" in case there is an element y in the algebra such that*

$$x+xy+y=0, \tag{1}$$

and then y will be called the "quasi-inverse" of x.

Since $(1+x)(1+y)=1+x+xy+y$, we see at once that if an element x of A is quasi-regular in A, then $1+x$ is regular in A'; and conversely, if $1+x$ is regular in A' for x in A, then x is quasi-regular in A', and actually in A as the following result shows.

LEMMA 1. *Let A be an algebra over F. If $A=A'$, a quantity of A is regular if and only if it has the form $1+x$ where x is quasi-regular. If $A\neq A'$, a quantity of A' is regular if and only if it is expressible as $\alpha(1+x)$ where α is a nonzero element of F and x is a quasi-regular element of A.*

[2] For, if $[b]$ is any regular element of $A-R$, then $[b][c]=[1]$, $bc=1+r$ with r in R. Since we have already proved $R\leqq H$ it follows that $1+r$ is regular, bc is regular, so that b is regular.

The case $A=A'$ is completed by the remark above the statement of the lemma. In the case $A\neq A'$ the same remark shows that if x is quasi-regular in A, then $1+x$ is regular and so is $\alpha(1+x)$. Conversely, let $g=\alpha+x_0$ be any regular element of A' with α in F and x_0 in A. We readily find that $\alpha\neq 0$ so that $g=\alpha(1+x)$, $1+x$ is regular, and

$$(1 + x)(\beta + y) = \beta + (\beta x + xy + y) = 1$$

for some β in F and y in A. Since $\beta x+xy+y$ is in A and β is in F, we must have $\beta=1$, $x+xy+y=0$, so that x is quasi-regular.

Observe that this lemma provides unique expressions for the regular quantities of A'.

If x is quasi-regular in A, its quasi-inverse is the unique element y such that $1+y$ is the inverse of $1+x$. Moreover,

$$(1+x)(1+y)=1=(1+y)(1+x),$$

whence $xy=yx$. Finally, the inverse $1+y$ of the regular element $1+x$ is known to be a polynomial in $1+x$ so that y is a polynomial in x.

LEMMA 2. *If x is a quasi-regular element of an algebra A, its quasi-inverse is unique, is a polynomial in x, and commutes with x.*

We now obtain the following main criterion.

THEOREM 2. *Let A be an algebra over a field F. Then an element r of A is in the radical of A if and only if $x+\alpha r$ is quasi-regular in A for every x of A which is quasi-regular and every α of F.*

If x is quasi-regular and r is in the radical, αr is in the radical of both A and A', $1+x$ is regular in A', and thus $1+x+\alpha r$ is regular by Theorem 1. By Lemma 1 the element $x+\alpha r$ of A must be quasi-regular. Conversely, suppose that r is an element of A with the property stated in the theorem. Any regular element g of A' has the form $g=\alpha+\alpha x$, $\alpha\neq 0$ in F, and x quasi-regular in A. By hypothesis $x+\alpha^{-1}r$ is quasi-regular so that

$$h=1+x+\alpha^{-1}r$$

is regular, and $\alpha h=g+r$ is regular. By Theorem 1 the element r is in the radical of A', hence in the radical of A.

A common characterization of the radical is that it consists of zero and all properly nilpotent elements. It may be noticed that this characterization is strongly in contrast with the present ones which are phrased in terms of addition rather than multiplication and are concerned with preserving regularity rather than nilpotency.

3. **Some applications.** As an application let us consider bound algebras which have been studied by M. Hall.[3] In his Theorem 3.6, Hall found that a bound algebra A contains three pairwise orthogonal idempotents e_1, e_2, e_3 with the following properties. If $A_i = e_iAe_i$ $(i=1, 2, 3)$ and R is the radical of A, then

(1) A is the supplementary sum $A = A_1 + A_2 + (A_3, R)$;

(2) $Re_1 = 0$, $e_2R = 0$;

(3) A_1 and A_2 are semi-simple.

We shall prove:

I. *If A has a unity element e, then $e = e_1 + e_2 + e_3$.*

For proof, let $e_0 = e - (e_1 + e_2 + e_3)$ so that e_0 is either zero or an idempotent orthogonal to each e_i. By property (1) we have $e_0 = a_1 + a_2 + (a_3 + r)$ with a_i in A_i, r in R, and we may always assume, without loss of generality, that either $a_3 = 0$ or else a_3 is not in R. By the orthogonality of the e_i and property (2) we have $e_0e_1 = 0 = a_1$, $e_2e_0 = 0 = a_2$. Now $e_0 = a_3 + r$, $e_0e_3 = 0 = a_3 + re_3$, $a_3 = -re_3$ in R so that $a_3 = 0$. Then $e_0 = r$ in R so that e_0 cannot be idempotent and we have $e_0 = 0$.

Next we again assume that A has a unity element and prove:

II. *If $x = a_1 = a_2 + (a_3 + r)$ is any element of A, a_i in A_i, r in R, then x is regular if and only if each a_i is regular in A_i.*

Since r is in R, x is regular if and only if $y = a_1 + a_2 + a_3$ is regular. If $z = b_1 + b_2 + (b_3 + s)$ is an element of A, b_i in A_i, s in R, then

$$yz = a_1b_1 + a_2b_2 + a_3b_3 + s_1, \qquad s_1 \text{ in } R,$$

and $yz = e = e_1 + e_2 + e_3$ if and only if $a_1b_1 = e_1$, $a_2b_2 = e_2$, $a_3b_3 = e_3 - s_1$. Hence a_1 and a_2 must be regular in their respective algebras. The third equation above shows that s_1 must be in A_3 as well as in R, hence in the radical of A_3. Then $e_3 - s_1$ is a regular element of A_3, a_3b_3 is regular in A_3, so that a_3 likewise must be regular in A_3. Conversely, if each a_i has an inverse b_i in A_i, it is clear that $b_1 + b_2 + b_3$ is the inverse of $a_1 + a_2 + a_3$.

An analogue of II not requiring a unity element will now be obtained.

III. *Let $x = a_1 + a_2 + (a_3 + r)$ be an element of a bound algebra A. Then x is quasi-regular if and only if each a_i is quasi-regular in A_i.*

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[3] *The position of the radical in an algebra*, Transactions of this Society, vol. 48 (1940), pp. 391–404. A "bound algebra" A with radical R is an algebra with the property that if $xR = Rx = 0$ for x in A, then x is in R. Hall has reduced the structure theory of arbitrary algebras to that of bound algebras and semi-simple algebras.

By Theorem 2 the element x is quasi-regular if and only if $x-r=y=a_1+a_2+a_3$ is quasi-regular. If each a_i has a quasi-inverse b_i in A_i, then $z_0=b_1+b_2+b_3$ has the property that $yz_0=a_1b_1+a_2b_2+a_3b_3$ so that $y+yz_0+z_0=0$, and y is quasi-regular. Conversely, suppose that y has a quasi-inverse $z=b_1+b_2+(b_3+s)$ with s in R and b_i in A_i. Then

$$y + yz + z = 0 = \sum_{i=1}^{3} (a_i + a_ib_i + b_i) + ys + s, \tag{2}$$

$$a_3 + a_3b_3 + b_3 + ys + s = 0, \qquad a_i + a_ib_i + b_i = 0 \tag{3}$$

for $i=1, 2$. Thus a_1 and a_2 are quasi-regular in A_1 and A_2, respectively, and the quantity

$$c = a_3 + a_3b_3 + b_3 = - s - ys \tag{4}$$

is in A_3, $e_3c=c=-e_3s-e_3ys=-e_3s-a_3s=-(e_3+a_3)s$. Since y commutes with its quasi-inverse (Lemma 2), we have $y+zy+z=0$ and as in (4) we are led to the equation

$$c' = a_3 + b_3a_3 + b_3 = - s - sy. \tag{5}$$

Hence $c'=c'e_3=-s(e_3+a_3)$. We have proved

$$c = - (e_3 + a_3)s, \qquad c' = - s(e_3 + a_3). \tag{6}$$

Now $(e_3+a_3)(e_3+b_3)=e_3+c=e_3-(e_3+a_3)s$ so that

$$(e_3 + a_3)(e_3 + b_3 + s) = e_3. \tag{7}$$

Likewise, by forming $(e_3+b_3)(e_3+a_3)=e_3+c'$ we find

$$(e_3 + b_3 + s)(e_3 + a_3) = e_3. \tag{8}$$

Since e_3 is the unity element of A_3, the results (7) and (8) show that the element e_3+a_3 of A_3 is not a divisor of zero in A_3. Thus e_3+a_3 is a regular element of A_3 so that a_3 must be quasi-regular. This completes the proof.

One may observe that property (3) is an immediate consequence of the fact III and Theorem 2. Suppose that r_1 is in the radical of A_1 and $x=a_1+a_2+(a_3+r)$ is any quasi-regular element of A. Then a_1 is quasi-regular in A_1, $a_1+\alpha r_1$ is quasi-regular in A_1 for every α of F, and $x+\alpha r_1=(a_1+\alpha r_1)+a_2+(a_3+r)$ has the requisite form for a quasi-regular element of A. By Theorem 2 the quantity r_1 is in R as well as in A_1. But then $r_1=0$ by property (1). Thus A_1 is semi-simple, and similarly A_2 is semi-simple.

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