BOOK REVIEWS

Diophantische Gleichungen. By T. Skolem. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 5, Heft 4. Berlin, Springer, 1938. 130 pp. R.M. 15.

Included in the author's preface is this statement, "Hier dagegen ist es versucht worden, eine zusammenfassende Darstellung der Theorie der diophantischen Gleichungen zu geben von den einfachsten bis zu den schwierigsten, die man bis jetzt hat bewaltigen können, aber nur soweit sie allgemeinen Methoden zugänglich sind. Es gibt bekanntlich auf diesem Gebiete sehr viele Untersuchungen über ganz spezielle Gleichungen; auf solche wird nicht eingegangen."

All the equations considered by Skolem, with the exception of several types considered in Chapter II, are of the form

$$f(x_1, x_2, \cdots, x_n) = 0,$$

the left-hand member of the equation being a polynomial in the unknowns $x_1, x_2, \cdots, x_n$ with given integral coefficients, the problem being to find all rational solutions or, in particular, integral solutions. Obviously, not all equations of this type which have been treated in the literature could have been considered in a pamphlet of this size. Hence he has elected, as he states above, to take up mainly what he regards as the most general types of equations concerning which definite results have been found. For example, congruences are special kinds of diophantine equations, but Skolem has wisely omitted any discussion of them, except for certain systems considered on two pages in the first chapter.

Many theorems are stated and proofs of some are given. When the proofs are not given references are usually indicated which enable us to find the demonstrations in the literature.

Diophantine analysis is noteworthy for the great interest which mathematicians have exhibited in certain methods which so far have been applied only to comparatively special equations. A number of these are not mentioned by Skolem, and we think it will be illuminating to refer to them in the course of this review.

In the first chapter of the book, the author considers linear equations of the form

$$\sum_{s=1}^{n} a_{rs}x_s = b_r,$$

$r=1, 2, \cdots, m$, and gives a number of the results of Heger, Smith and Frobenius. This work connects up with the theory of the elementary divisors of a matrix.

In Chapter II Skolem treats equations which are linear in each unknown, in particular the bilinear equation

$$\sum_{rs} a_{rs}x_r y_s = 0.$$

In this general theory it is convenient to employ equations in determinant form.

The third chapter treats quadratic equations in two or more unknowns. The homogeneous case is treated first. Smith's criterion for the integral solution of

$$ax^2 + a_1x_1^2 + a_2x_2^2 + 2bx_1x_2 + 2b_1xx_2 + 2b_2xx_1 = 0$$

is given. Proceeding to the equation

$$\sum_{i=1}^{n} a_i x_i^2 = 0, \tag{1}$$

the beautiful theory for the case $n=3$ is treated. Meyer's results on (1) are stated and also this theorem that (1) for $n=5$ is always satisfied in integers not all zero with integers $a_1, a_2, \cdots, a_5$, not zero, and not all of like sign. The first complete proof is by Dickson.[1] Skolem gives Mordell's proof of this after first giving his own proof of Hasse's results concerning the congruence

$$\sum_{i=1}^{n} a_i x_i^2 \equiv 0 \pmod{m},$$

on which the latter shows the solution of (1) depends, if f is indefinite.

Next the nonhomogeneous quadratic equations are treated. He considers the classic theory of representation of integers by ternary forms, but does not mention a number of theories concerning quadratic forms which have so far attracted great attention from many mathematicians, such as the representation of integers as the sum of squares and closely related topics developed by Liouville, Eisenstein, Minkowski, Smith, Bell, Nasimoff, Hardy, Uspensky, Mordell and others, using the methods of elliptic functions, analytic additive number theory, for example. Also, the application of quaternions and related linear algebras to such questions as developed by A. Hurwitz, Dickson and others is not taken up.

[1] Studies in the Theory of Numbers, pp. 68–70.

In the fourth chapter entitled "Multiplikative Gleichungen" the author gives, with the proof, the results of Bell and Ward on the equations,

$$A_i x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_k^{a_{ik}} = B_i y_1^{b_{i1}} y_2^{b_{i2}} \cdots y_e^{b_{ie}}.$$

He then takes up equations obtained by expanding

$$N(\alpha_1 x_1 + \cdots + \alpha_n x_n)$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are integers in the algebraic field K of degree n, N denotes the norm in K, and setting

$$N = a, \tag{2}$$

where a is a rational integer, and treats by the use of units in K the problem of finding other sets of solutions of (2), having given one set, as well as methods for determining if there exists any set.

He then examines equations of the form

$$f(x_1, x_2, \cdots, x_n) = hy_1^{e_1} y_2^{e_2} \cdots y_m^{e_m},$$

where f is decomposable in some algebraic field. The results of Ward[2] on the equation

$$x^2 - dy^2 = z^m, \qquad d > 0,$$

are given. Ward obtains the primitive solutions of this by use of the field defined by $d^{1/2}$.

Dickson's solution of

$$q = ax^2 + bxy + cy^2 = w_1 \cdots w_n,$$

is mentioned, then a generalization to the equation

$$q = hz_1^{a_1} \cdots z_n^{a_n}$$

is discussed by Skolem. He remarks that the methods he employs extend to the equation,

$$f(x_1, \cdots, x_m) = hz_1^{a_1} \cdots z_n^{a_n}$$

where f is decomposable into m linear factors in some algebraic field, and where the degree of $f > 2$. For the case $h = 1$, $a_1 = \cdots = a_n = 1$, a solution involving the use of ideals was given by Wahlin.

The fifth chapter is concerned with the application of geometry to various types of diophantine equations, when we are seeking rational solutions. It is in this chapter, in the sixth and in the last chapter,

[2] Transactions of this Society, vol. 38 (1935), pp. 447–457.

that it appears the most general results are to be found in the work. The curves

$$f(x, y) = 0, \quad \text{or} \quad f(x, y, z) = 0$$

in homogeneous coordinates are classified for arithmetical purposes according to their genus, the coefficients of the equations being integral. Two curves are called equivalent when they are connected with each other by a birational transformation with rational coefficients. Thus two curves

$$f_1(x, y) = 0, \qquad f_2(\xi, \eta) = 0$$

are equivalent if the coordinates x, y are rationally expressible with rational coefficients in terms of ξ, η, and conversely.

If the curve is of genus zero, it was shown by Poincaré that it is equivalent to a straight line or a conic, from which the theory of the rational points on the original curve is easily derived.

For a curve of genus unity Poincaré also proved that if it has a rational point, it is equivalent to a curve of the third degree, and, in particular, the cubic is equivalent to a curve whose equation can be written in the Weierstrass normal form

$$y^2 = 4x^3 - g_2x - g_3. \tag{3}$$

The latter statement was proved by Mordell in 1912.

There is a well known geometric method for finding rational points on a curve when certain rational points are known initially. If one of the latter points is P (not a point of inflexion), the tangent at P meets the curve in another point P_1 whose coordinates are determined by an equation of the first degree and hence are rational. If Q is another such point, the chord PQ will meet the cubic in a rational point R, in general different from P.

To interpret this process analytically Poincaré and Hurwitz noted that the equation (3) could be given in parametric form by the use of Weierstrass elliptic p functions so that

$$x = p(u), \qquad y = p'(u). \tag{3a}$$

It was proved by Mordell by the use of such ideas that all the rational points on (3) could be found from a finite number of rational points, so that, using (3a), if $u_1, u_2, \cdots, u_n$ satisfy (2) all points are given by $m_1u_1+m_2u_2+ \cdots +m_nu_n$, where the m's are integers, that is, the parametric values for all the rational points form an additive Abelian group with a finite basis.

Skolem mentions the work of Fueter and Brunner on the equation

$$x^3 - y^2 = D.$$

They were able to determine the number of solutions of this equation for various values of D, using the field defined by $(-D)^{1/2}$. The author then considers curves of genus $p > 1$ and notes that it has been conjectured that such curves contain only a finite number of rational points. A curve of this type is given by the Fermat equation

$$x^n + y^n = z^n.$$

In this connection he mentions only the work of Kummer, Wieferich, Mirimanoff, Furtwangler, and Kapferer, on Fermat's last theorem.

A rational system of points is defined as a finite system of points such that all the symmetric functions of their parametric values are rational. Using this idea Weil showed that Mordell's result concerning the parametric values associated with the rational points on a cubic curve forming an additive Abelian group could be extended to curves of genus $p > 1$.

In Chapter VI Skolem goes into detail concerning the theorems associated with the names of Thue and Siegel. These theorems enable us to state that certain equations of the type

$$f(x, y) = c,$$

have only a finite number of solutions. These ideas seem to have had their beginnings in the work of Runge (1887), who proved among other results the following one:

Let $f(x, y)$ be a polynomial with integral coefficients of degree n irreducible in the rational field, the homogeneous part of f of highest degree not being the power of an irreducible polynomial. Then $f(x, y) = c$ has only a finite number of solutions in rational integers.

Thue's theorem of the year 1909 has to do with a polynomial of a type a bit different from that treated in Runge's theorem. It is as follows:

If $f(x, y)$ is a homogeneous irreducible polynomial of degree greater than 2 with integral coefficients and c is a given integer not equal to 0, *the equation $f(x, y) = c$ has only a finite number of integral solutions.*

The author proves this theorem as well as a number of lemmas concerning the approximation to a real algebraic number by the use of a rational number, on which these investigations depend. Using these results it is possible to show that the equation

$$ay^2 + by + c = dx^n,$$
$$a(b^2 - 4ac)d \neq 0, \qquad n \geqq 3,$$

has only a finite number of solutions as well as the equation

$$y^2 = ax^n + bx^{n-1} + \cdots + k$$

where, in the latter, the right-hand member has at least three different zeros.

After considering a number of special equations of the form

$$x^n + dy^n = \pm 1$$

where $n=3$ or 4, Skolem applies the theory of p-adic numbers to the equation

$$N(\alpha x + \beta y + \gamma z) = h$$

where α, β and γ are integers in an algebraic field K of degree n. He finds equations of this type which have only a finite number of solutions for $n=5$.

We now signalize a problem which seems fundamental in this subject. If we consider the irreducible equation

$$f(x_1, x_2, \cdots, x_k) = c$$

where f is of degree n with integral coefficients and with c integral and also f homogeneous we know from the theory of units in an algebraic field that for $k=n$ and $c=1$, there exist equations of this type with an infinity of integral solutions. On the other hand, if $k=2$, $n>2$, Thue's theorem states that there cannot be more than a finite number of solutions. The question is, how far must k be increased to obtain equations of this type with an infinity of solutions? If $n=3$, we have $k=3$.

The arithmetical theory of Hermitian forms is not considered, likewise Waring's theorem. It is not exactly surprising that the latter topic has been omitted, as it would merit a volume in itself.

Skolem has written a very interesting book. It is surprising how much arithmetical meat he has packed into the space he employs.

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Sur les Fonctions Orthogonales de Plusieurs Variables Complexes, avec les Applications à la Théorie des Fonctions Analytiques. By Stefan Bergman. New York, Interscience Publishers, 1941. 62 pp. $1.50.

This book was to appear as one of the Mémorial des Sciences Mathématiques series, but circumstances were such that the edition reached only the galley proof stage. The book is a photostatic edi-