ZERO-DIMENSIONAL FAMILIES OF SETS[1]

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A family $\Phi=\{A_\alpha\}$ of subsets of a topological space X will be called 0-*dimensional* if given an open set U such that $A_{\alpha_0}\subset U$, there is an open set V such that (1) $A_{\alpha_0}\subset V\subset U$ and (2) $(\overline{V}-V)\sum_\alpha A_\alpha=0$. We enumerate below a few of the most common 0-dimensional families. In each case the proof of 0-dimensionality is easy, and is therefore omitted.

(I) Every family of disjoint open subsets of a topological space is 0-dimensional.

(II) Let Y be a locally connected subset of a topological space X. The family Φ of the components of Y is 0-dimensional.

(III) Let Y be a compact and closed subset of a metric space X. The family Φ of the components of Y is 0-dimensional.

(IV) Let Y be a subset of a metric space X. The family Φ consisting of the individual points of Y is 0-dimensional if and only if $\dim Y=0$.

(V) Let Φ be a family of closed subsets of a compact metric space X. If, given any sequence $F, F_1, F_2, \cdots$ of sets of Φ, the relation $F\cdot\liminf F_i\neq 0$ implies $\liminf F_i\subset F$, then the family Φ is called *upper-semi-continuous*. In this case the sets of the family Φ are disjoint. There is a standard way of introducing a topology into the family Φ which leads to a separable metrizable *hyperspace* Φ^*. The family Φ is 0-dimensional if and only if $\dim \Phi^*=0$. In particular, Φ is 0-dimensional whenever it is upper-semi-continuous and countable.

(VI) Let Y be a subset of a topological space X and let Y be homeomorphic with a subset of the linear continuum. The family Φ of the components of Y is 0-dimensional.

The purpose of this note is to establish the following theorem:

THEOREM. *Let X be a unicoherent Peano continuum,*[2] $\Phi=\{A_\alpha\}$ *a 0-dimensional family of subsets of X, and x_1 and x_2 two points of X. If none of the sets A_α cuts X between x_1 and x_2,*[3] *then $\sum_\alpha A_\alpha$ does not cut X between x_1 and x_2.*

Various corollaries can be obtained by taking X to be the n-sphere

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[1] Presented to the Society, December 26, 1939, under the title *On 0-dimensional upper-semi-continuous collections.*

[2] A Peano continuum X is called *unicoherent* if given any decomposition $X=X_1+X_2$ into continua, the set $X_1\cdot X_2$ is a continuum.

[3] A set $A\subset X$ *cuts* X between x_1 and x_2 if $X-A$ contains no continuum joining x_1 and x_2.

S^n $(n>1)$ or the n-cube Q^n $(n>0)$ and Φ to be one of the families (I)–(VI).[4]

We shall establish two lemmas before giving the proof of the theorem.

LEMMA 1. *Let X be a unicoherent Peano continuum. Let A_1 and A_2 be open and disjoint subsets of X, and let A_2 be connected. If neither A_1 nor A_2 cuts X between the points x_1 and x_2, then A_1+A_2 does not cut X between x_1 and x_2.*

PROOF. Let C_2 be the component of $X-A_2$ which contains x_1 and x_2. Let $B_1=A_1\cdot C_2$ and $B_2=X-C_2$. It follows easily that B_1 and B_2 are open, $B_1\cdot B_2=0$, B_2 and $X-B_2=C_2$ are connected, and $A_1+A_2 \subset B_1+B_2$. It will be sufficient to show that B_1+B_2 does not cut X between x_1 and x_2.

Since $B_1\subset A_1$, the set B_1 does not cut X between x_1 and x_2. We will denote the component of $X-B_1$ which contains x_1 and x_2 by C_1. If $C_1\cdot B_2=0$, then $C_1\subset X-(B_1+B_2)$, so that B_1+B_2 does not cut X between x_1 and x_2. We will suppose then that $C_1\cdot B_2\neq 0$. Since C_1 and B_2 are connected it follows that C_1+B_2 is connected. But $C_1+B_2\subset X-B_1$. Hence $C_1+B_2\subset C_1$, and therefore $B_2\subset C_1$. Since $C_2=X-B_2$, it follows that $X=C_1+C_2$. Since C_1 and C_2 are continua and X is unicoherent, it follows that $C_1\cdot C_2$ is a continuum. But x_1 and x_2 belong to $C_1\cdot C_2$, and $C_1\cdot C_2\subset X-(B_1+B_2)$. Hence B_1+B_2 does not cut X between x_1 and x_2.

LEMMA 2. *Let X be a unicoherent Peano continuum and let $A_1, A_2, \cdots, A_n, \cdots$ be a sequence of disjoint open subsets of X. If none of the sets A_n cuts X between x_1 and x_2, then $\sum_{n=1}^{\infty} A_n$ does not cut X between x_1 and x_2.*

PROOF. Since X is locally connected and separable, every open set in X consists of a countable number of components each of which is open. It is clear, then, that we may assume each set A_n to be connected.

Let k be any positive integer. Using Lemma 1, it follows by finite induction that $A_1+A_2+\cdots+A_k$ does not cut X between x_1 and x_2. Let C_k be the component of $X-(A_1+A_2+\cdots+A_k)$ which contains x_1 and x_2. Then $\prod_{k=1}^{\infty} C_k$ is a continuum containing x_1 and x_2, and $\prod_{k=1}^{\infty} C_k \subset X-\sum_{k=1}^{\infty} A_k$. Hence $\sum_{k=1}^{\infty} A_k$ does not cut X between x_1 and x_2.

[4] See R. L. Moore, Proceedings of the National Academy of Sciences, vol. 10 (1934), p. 356, and S. Eilenberg, Fundamenta Mathematicae, vol. 26 (1936), pp. 76–77.

We now return to the *proof of the theorem*:

For any F in Φ there is an open set $U(F)$ such that $F \subset U(F)$, and $U(F)$ does not cut X between x_1 and x_2. Since Φ is 0-dimensional, there is an open set $V(F)$ such that $F \subset V(F) \subset U(F)$ and $[\overline{V(F)} - V(F)]\sum F = 0$. By the Lindelöf covering theorem, there is a sequence $F_1, F_2, \cdots$ of elements of Φ such that $\sum F \subset \sum_{i=1}^{\infty} V(F_i)$. Now let

$$A_1 = V(F_1), \qquad A_2 = V(F_2) - \overline{V(F_1)},$$
$$A_k = V(F_k) - \overline{[V(F_1) + \cdots + V(F_{k-1})]}, \cdots .$$

The sets $A_1, A_2, \cdots, A_k, \cdots$ are open and disjoint, and no one of them cuts X between x_1 and x_2. But, as is easily shown, $\sum F \subset \sum_{k=1}^{\infty} A_k$. Hence in view of Lemma 2, $\sum F$ does not cut X between x_1 and x_2.

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SUMS OF FOURTH POWERS OF GAUSSIAN INTEGERS

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It is the purpose of this note to give necessary and sufficient conditions for the expressibility of a Gaussian integer as a sum of fourth powers of Gaussian integers; and then to determine an upper bound to the number of fourth powers necessary when the conditions are satisfied. Our results are as follows:

THEOREM. *A Gaussian integer is expressible as a sum of fourth powers of Gaussian integers if and only if its imaginary coordinate is divisible by* 24. *Every integer* $a+24bi$, *where* a *and* b *are rational integers, is expressible as a sum of* 18 *or fewer fourth powers.*

First we prove that the condition is necessary. We note that[1]

$$(1) \qquad (x + yi)^4 = x^4 - 6x^2y^2 + y^4 + 4ixy(x^2 - y^2).$$

It is obvious that $xy(x^2-y^2)$ is divisible by 2 and by 3. Hence any fourth power has an imaginary coordinate divisible by 24, and any sum of fourth powers has the same property.

The converse of this is included in the second statement in the theorem, which we now proceed to prove. The author[2] has shown that a Gaussian integer $a+2bi$ is expressible as a sum of two squares

[1] Latin letters will represent rational integers throughout this paper.

[2] *Integers of quadratic fields as sums of squares*, Transactions of this Society, vol. 48 (1940), p. 410, Theorem 2.