A NOTE ON FUNCTIONS OF EXPONENTIAL TYPE[1]

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An entire function $f(z)$ is said to be of exponential type at most T if

$$\limsup_{n\to\infty} |f^{(n)}(z)|^{1/n} \leqq T \tag{1}$$

for some z (and hence for every z, uniformly for z in any bounded set). An equivalent condition[2] is that for each positive ϵ

$$|f(z)| < e^{(T+\epsilon)|z|}$$

for all sufficiently large $|z|$. The following three theorems were proved respectively by D. V. Widder [**4**], I. J. Schoenberg [**2**], and H. Poritsky [**1**] and J. M. Whittaker [**3**].

THEOREM 1. (Widder.) *If a real function $f(z)$, of class C^∞ in $0 \leqq x \leqq 1$, satisfies the condition*

$$(-1)^n f^{(2n)}(x) \geqq 0, \qquad 0 \leqq x \leqq 1;\ n = 0, 1, 2, \cdots, \tag{2}$$

then $f(x)$ coincides on $(0, 1)$ with an entire function of exponential type at most π.

THEOREM 2. (Schoenberg.) *If $f(z)$ is an entire function of exponential type at most T, and if*

$$f^{(2n)}(0) = f^{(2n)}(1) = 0, \qquad n = 0, 1, 2, \cdots, \tag{3}$$

then $f(z)$ is a sine polynomial of order at most T/π:

$$f(z) = \sum_{k=0}^{N} a_k \sin k\pi z, \qquad N \leqq T/\pi.$$

Let $\Lambda_n(z)$ be the polynomial of degree $2n+1$ determined by the relations

$$\Lambda_0(z) = z; \qquad \Lambda_n(0) = \Lambda_n(1) = 0, \qquad n \geqq 1;$$
$$\Lambda_n''(z) = \Lambda_{n-1}(z), \qquad n \geqq 1.$$

THEOREM 3. (Poritsky-Whittaker.) *If $f(z)$ is an entire function of exponential type at most T, $T < \pi$, then $f(z)$ can be represented in the form*

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[2] G. Valiron, *Lectures on the General Theory of Integral Functions*, Toulouse, 1923, p. 41.

$$f(z) = \sum_{n=0}^{\infty} f^{(2n)}(1)\Lambda_n(z) - \sum_{n=0}^{\infty} f^{(2n)}(0)\Lambda_n(z-1), \tag{4}$$

where the series converge uniformly in any bounded region.

An expansion of the form (4) is called a Lidstone series.

The chief purpose of this note is to give a short proof of Theorem 1 which I obtained some time after Professor Widder communicated the theorem to me.[3] I also give simple proofs of Theorems 2 and 3, suggested by the proof of Theorem 1.

LEMMA 1. *If k and n are positive integers, and $f(x)$ is of class C^{2n} in $0 \leqq x \leqq 1$, then*

$$\begin{aligned}\int_0^1 f(x) \sin k\pi x\, dx = \sum_{m=0}^{n-1} \frac{(-1)^m}{(k\pi)^{2m+1}} \{f^{(2m)}(0) - (-1)^k f^{(2m)}(1)\} \\ + \frac{(-1)^n}{(k\pi)^{2n}} \int_0^1 f^{(2n)}(x) \sin k\pi x\, dx.\end{aligned} \tag{5}$$

This is proved by integrating the left-hand side repeatedly by parts.

LEMMA 2. *If $g(x)$ is non-negative and concave in $a \leqq x \leqq b$, and*

$$\int_a^b g(x)dx \leqq A,$$

then

$$g(x) \leqq \frac{2A}{b-a}, \qquad a \leqq x \leqq b. \tag{6}$$

Let $g(x)$ take its maximum, G, at $x = x_0$. Since $g(x)$ is concave, the graph of $g(x)$ is above the broken line connecting the points $(a, 0)$ and $(b, 0)$ to (x_0, G). The area under the broken line is $\frac{1}{2}G(b-a)$. Hence $\frac{1}{2}G(b-a) \leqq A$, and (6) follows.

PROOF OF THEOREM 1. Take $k=1$ in Lemma 1. By hypothesis the terms of the sum on the right of (5) are all non-negative. Hence, for any positive integer n,

$$B = \int_0^1 f(x) \sin \pi x\, dx \geqq \frac{1}{\pi^{2n}} \int_0^1 (-1)^n f^{(2n)}(x) \sin \pi x\, dx \geqq 0,$$

$$\int_\delta^{1-\delta} (-1)^n f^{(2n)}(x) \sin \pi x\, dx \leqq B\pi^{2n}, \qquad 0 < \delta < \tfrac{1}{2},$$

[3] (Added in proof.) Essentially the same proof was found independently by Professor Schoenberg.

$$\int_{\delta}^{1-\delta} (-1)^n f^{(2n)}(x)dx \leqq \frac{B\pi^{2n}}{\sin \pi\delta}\cdot$$

Since $(-1)^n f^{(2n+2)}(x) \leqq 0$, $(-1)^n f^{(2n)}(x)$ is non-negative and concave in $(\delta, 1-\delta)$. By Lemma 2,

$$(7) \qquad 0 \leqq (-1)^n f^{(2n)}(x) \leqq \frac{2B\pi^{2n}}{(1-2\delta)\sin \pi\delta}, \qquad \delta \leqq x \leqq 1-\delta.$$

Let $h=\frac{1}{2}-\delta$. For any x in $(\delta, 1-\delta)$, Taylor's theorem with remainder of order 2, applied to $f^{(2n)}(x)$, leads to the relation

$$f^{(2n+1)}(x) = \pm \frac{1}{h}\left[f^{(2n)}(x \pm h) - f^{(2n)}(x)\right] \mp \tfrac{1}{2}hf^{(2n+2)}(x \pm \theta h), \qquad 0 < \theta < 1,$$

where the upper signs or the lower signs are taken according as $x+h$ or $x-h$ is in $(\delta, 1-\delta)$. Using (7), written both for n and $n+1$, we obtain

$$\left| f^{(2n+1)}(x) \right| \leqq \left(\frac{2}{h} + \frac{\pi^2 h}{2}\right)\frac{2B\pi^{2n}}{(1-2\delta)\sin \pi\delta} = C(\delta)B\pi^{2n+1},$$

where $C(\delta)$ depends only on δ. Thus

$$(8) \qquad \limsup_{n\to\infty} \left| f^{(n)}(x) \right|^{1/n} \leqq \pi,$$

uniformly in any interval $\delta<x<1-\delta$, where $0<\delta<\frac{1}{2}$. The relation (8) implies first that the Taylor series of $f(x)$ about any point in $(0, 1)$ converges to an entire function coinciding with $f(x)$ in $(0, 1)$; (8) then shows that this function is of exponential type at most π.

PROOF OF THEOREM 2. Since $f(x)$ is of class C^∞, it is represented in $(0, 1)$ by its Fourier sine series:

$$(9) \qquad f(x) = \sum_{k=0}^{\infty} a_k \sin k\pi x, \qquad 0 < x < 1,$$

where

$$a_k = 2\int_0^1 f(t) \sin k\pi t\, dt.$$

Since (3) is true, Lemma 1 shows that

$$a_k = \frac{2(-1)^n}{(k\pi)^{2n}} \int_0^1 f^{(2n)}(t) \sin k\pi t\, dt$$

for every positive integer n. Hence

$$|a_k| \leqq \frac{2}{(k\pi)^{2n}} \max_{0\leqq x\leqq 1} |f^{(2n)}(x)|. \tag{10}$$

If $k\pi > T$, for large n we have

$$\max_{0\leqq x\leqq 1} |f^{(2n)}(x)| \leqq S^{2n}, \qquad T < S < k\pi,$$

since $f(x)$ is of exponential type at most T. From (10) it then follows that $a_k = 0$ if $k\pi > T$. Thus all terms of (9) with $k > T/\pi$ vanish, and Theorem 2 is proved.

Proof of Theorem 3. The function $f(x)$ is represented in (0, 1) by the Fourier series (9), and $\frac{1}{2}a_k$ is just the integral on the left of (5). Hence, for every positive integer n,

$$\begin{aligned} f(x) = {} & \sum_{m=0}^{n-1} f^{(2m)}(1) \sum_{k=1}^{\infty} \frac{2(-1)^{k+m+1} \sin k\pi x}{(k\pi)^{2m+1}} \\ & - \sum_{m=0}^{n-1} f^{(2m)}(0) \sum_{k=1}^{\infty} \frac{2(-1)^{m+1} \sin k\pi x}{(k\pi)^{2m+1}} + \tfrac{1}{2} R_n(x), \end{aligned} \tag{11}$$

where

$$R_n(x) = (-1)^n \int_0^1 f^{(2n)}(t) \sum_{k=1}^{\infty} \frac{\sin k\pi x \sin k\pi t}{(k\pi)^{2n}} dt.$$

The infinite series appearing in (11) are the Fourier sine series of $\Lambda_m(x)$ and $\Lambda_m(1-x)$, as given by Whittaker;[4] they can easily be checked by successive integrations of the well known Fourier sine series of $\Lambda_0(x) = x$. Relation (11) becomes

$$f(x) = \sum_{m=0}^{n-1} f^{(2m)}(1)\Lambda_m(x) - \sum_{m=0}^{n-1} f^{(2m)}(0)\Lambda_m(x-1) + R_n(x).$$

We have thus obtained a "Lidstone series with remainder," with an expression for the remainder as a real integral equivalent to that given by Widder [4].

We have

$$|R_n(x)| \leqq \frac{1}{\pi^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^{2n}} \int_0^1 |f^{(2n)}(t)| \, dt,$$

and this approaches zero as $n \to \infty$, uniformly in (0, 1), if $f(z)$ satisfies (1) with $T < \pi$. Hence under the hypothesis of Theorem 3 the series in (4) converges to $f(x)$ uniformly in (0, 1). Now Whittaker has

[4] [3, p. 454].

shown[5] that if a Lidstone series converges for some non-integral value of z, it converges for all z, uniformly in any bounded region, and so represents an entire function, which in our case must be $f(z)$. This completes the proof of Theorem 3.

By applying, instead of Lemma 1, the formula obtained by integrating

$$\int_0^1 f(x) \cos \tfrac{1}{2}k\pi x \, dx$$

by parts, we can prove Schoenberg's theorem [2], analogous to Theorem 2, that a function $f(z)$ of exponential type is a cosine polynomial if $f^{(2n)}(1)=f^{(2n+1)}(0)=0$ $(n=0, 1, 2, \cdots)$; and we can obtain Whittaker's result [3] corresponding to Theorem 3, concerning the expansion of $f(z)$ in a series with coefficients $f^{(2n)}(1), f^{(2n+1)}(0)$. The analogue of Theorem 1 is

THEOREM 4. *If $f(x)$ is of class C^∞ in $0 \leqq x \leqq 1$, and*

$$f^{(4n)}(x) \geqq 0, \quad (-1)^n f^{(2n)}(1) \geqq 0, \quad (-1)^n f^{(2n+1)}(0) \leqq 0, \qquad n = 0, 1, 2, \cdots,$$

then $f(x)$ coincides over $(0, 1)$ with an entire function of exponential type at most $\pi/2$.

REFERENCES

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3. J. M. Whittaker, *On Lidstone's series and two-point expansions of analytic functions*, Proceedings of the London Mathematical Society, (2), vol. 36 (1933–1934), pp. 451–469.

4. D. V. Widder, *Functions whose even derivatives have a prescribed sign*, Proceedings of the National Academy of Sciences, vol. 26 (1940), pp. 657–659.

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[5] [3, p. 455].