THE FINITE DIFFERENCES OF POLYGENIC FUNCTIONS[1]

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By a polygenic function $f(z)$ we shall mean a function analytic in x and y separately, but whose real and imaginary parts are not required to satisfy the Cauchy-Riemann equations. At any point z the derivative of such a function will depend on θ, the angle at which the incremented point (used in defining the derivative) approaches z. The set of these numbers, for a fixed z, but for different θ, form a circle. The equation for the derivative was given by Riemann in his classic dissertation (1851), but Kasner was the first to point out that it was a circle and make a detailed study of its geometry.[2] Hedrick called it the Kasner circle.

In this paper we shall be concerned with the finite difference quotients of polygenic functions. We shall show how a surface can be constructed for each point z representing the difference quotient, and the derivative circle is a cross section of this surface.

The conjugate form. Regard

$$z = x + iy, \qquad \bar{z} = x - iy$$

as a linear substitution, and perform its inverse

$$x = \frac{1}{2}(z + \bar{z}), \qquad y = \frac{1}{2i}(z - \bar{z})$$

on $f(z)$. The resulting $F(z, \bar{z})$ will be called the conjugate form of f. Let $D_z F$ and $D_{\bar{z}} F$ be the partial derivatives[3] of $F(z, \bar{z})$, regarding z and $\bar{z}$ as independent variables. That is,

$$D_z F = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \tfrac{1}{2}(D_x - D_y)f, \tag{1}$$
$$D_{\bar{z}} F = \tfrac{1}{2}(D_x + D_y)f.$$

The operator E^{ω}. Let $\omega = \rho e^{i\theta}$. We define

$$E^{\omega} f(z) = f(z + \omega).$$

[1] Presented to the Society, February 25, 1939, under the title *A geometric interpretation of the difference quotient of polygenic functions.*

[2] *General theory of polygenic or non-monogenic functions*; *The derivative congruence of circles*, Proceedings of the National Academy of Sciences, vol. 13 (1928), pp. 75–82. *A new theory of polygenic functions*, Science, vol. 66 (1927). Also, *The Geometry of Polygenic Functions*, Kasner and DeCicco—a book in the course of preparation.

[3] In Kasner's notation, these are $\mathfrak{M}(f)$ and $\mathfrak{P}(f)$.

More precisely, this means

$$E^{\omega}f(x, y) = f(x + \rho \cos \theta, y + \rho \sin \theta).$$

E may also have a "partial" meaning:

$$E_x^a f(x, y) = f(x + a, y).$$

And we see the equivalence of the two operators:

$$E^{\omega} = E_x^{\rho \cos \theta} \cdot E_y^{\rho \sin \theta}. \tag{2}$$

Taylor's expansion may be written in the form

$$E_x^a f(x, y) = \left(1 + aD_x + \frac{1}{2!} a^2 D_x^2 + \cdots \right) f(x, y) = \exp(aD_x) f. \tag{3}$$

Now, by combining (2) and (3) we obtain the operational equivalence

$$E^{\omega} = \exp(\rho \cos \theta \, D_x + \rho \sin \theta \, D_y).$$

$(\cos \theta \, D_x + \sin \theta \, D_y) f$ is nothing more than the directional derivative of f, which we designate by $\mathcal{D}f$. Substituting from (1) for D_x and D_y, we see that

$$\mathcal{D} = e^{i\theta} D_z + e^{-i\theta} D_{\bar{z}}, \tag{4}$$

$$E^{\omega} = \exp(\rho \mathcal{D}). \tag{5}$$

The differential quotient. We define

$$_{\omega}\Delta f(z) = \frac{f(z + \omega) - f(z)}{\omega},$$

whence, operationally,

$$_{\omega}\Delta = \frac{1}{\omega}[E^{\omega} - 1].$$

Expanding by means of (4) and (5)

$$_{\omega}\Delta f(z) = \frac{1}{\rho e^{i\theta}}\left[\rho \mathcal{D} + \frac{\rho^2 \mathcal{D}^2}{2!} + \cdots \right] f(z)$$

$$= \left[D_z + e^{-2i\theta} D_{\bar{z}} + \frac{\rho}{2!}(e^{i\theta} D_z^2 + 2e^{-i\theta} D_z D_{\bar{z}} + e^{-3i\theta} D_{\bar{z}}^2) + \cdots \right] f(z). \tag{6}$$

To obtain the derivative we let $\rho \to 0$. The resulting expression

$$[D_z + e^{-2i\theta} D_{\bar{z}}] f(z),$$

which Kasner calls γ, is immediately seen to be the points of a circle, when z is fixed and θ varies.

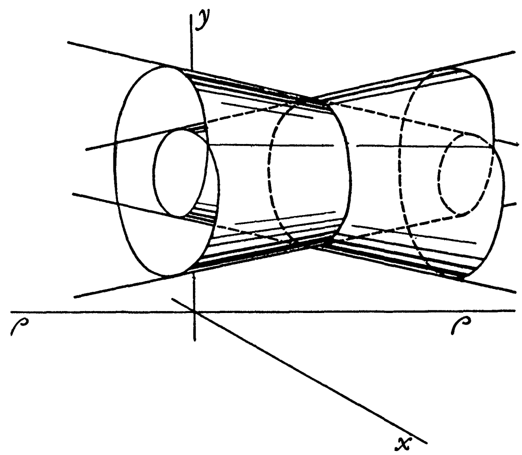

FIG. 1

For a geometric interpretation of (6), let us add a third coordinate ρ to the x, y plane. In this 3-space, (6) represents a surface for each fixed value of z. Such a surface we will call a Kasnercoid or K-coid of the function.

Example. $f(z) = x^2 + y^2 + i(x+y)$. Its conjugate form is $z\bar{z} + \frac{1}{2}(i+1)z + \frac{1}{2}(i-1)\bar{z}$. Using (6) we obtain for the difference quotient:

$$_{\omega}\Delta f = \bar{z} + \tfrac{1}{2}(i + 1) + e^{-2i\theta}[z + \tfrac{1}{2}(i - 1)] + \rho e^{-i\theta}.$$

The K-coid of a typical point is sketched in Figure 1.

This of course is a very simple case. In regard to the nature of K-coids in general, we state a theorem.

A curve will be called a doubler of a circle when it has the following properties:

1. In traversing the doubler once, we pass around the center of the circle twice.
2. No circle intersects the doubler in more than six points.
3. Every point on the circle is the midpoint of two points on the doubler.

(See Figure 2.)

THEOREM. *If ρ^2 be neglected, then sections of the K-coid sufficiently near the derivative circle are doublers of the derivative circle.*

Compare with the monogenic case:

For a monogenic function, the derivative circle reduces to a point and the nearby sections are nearly circles.

The purpose of this paper is to throw some light on the nature of derivatives. We may consider the K-coid compressed along the ρ-axis

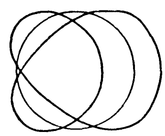

FIG. 2

so as to be a two-sheeted (at least for small ρ) Riemann surface with the derivative circle acting as "branch circle." Let us consider the familiar expression:

$$\lim_{\omega \to 0} \frac{f(z + \omega) - f(z)}{\omega},$$

and suppose $z+\omega$, in approaching z, travels along some curve. To each point on the curve correspond two points on the K-coid which approach a common point on the derivative circle.

Connection with the second derivative. Let us write (6) in the form

$$[e^{-i\theta}\mathfrak{D} + (\omega/2!)e^{-2i\theta}\mathfrak{D}^2 + \cdots]f(z).$$

Is the coefficient of $\omega/2!$ the second derivative of $f(z)$? This depends on what is meant by the second derivative, as there are several alternative methods of defining it.[4] According to Kasner's method,[5] the second derivative depends not only on the slope of the path of the incremented point but also on the curvature of the path. The expression is

$$[e^{-2i\theta}\mathfrak{D}^2 - 2ie^{-3i\theta}\kappa D_{\bar{z}}]f(z),$$

where κ is the curvature.

[4] See my unpublished Master's Essay, Columbia University.

[5] *The second derivative of a polygenic function*, Transactions of this Society, vol. 30, no. 4.

The corresponding expression for what I call the type A derivative—based on another, but equally logical definition—is merely the first term of the above expression.

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ON THE ASYMPTOTIC LINES OF A RULED SURFACE

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Many mathematicians have studied the surfaces every *asymptotic curve* of which belongs to a linear complex. I will here be content with the results given on pages 112–116 and 266–288 of a treatise[1] written by myself and Professor A. Cech. This treatise gives (p. 113) a very simple proof of the following theorem:

If every non-rectilinear asymptotic curve of a ruled surface S belongs to a linear complex, all these asymptotic curves are projective to each other.

We will find all the ruled surfaces, the non-rectilinear asymptotic curves of which *are projective to each other*, and prove conversely that *every one of these asymptotic curves belongs to a linear complex*. If c, c' are two of these asymptotic curves and if A is an arbitrary point of c, we can find on c' a point A' such that the straight line AA' is a straight generatrix of S. The projectivity, which, according to our hypothesis, transforms c into c', will carry A into a point A_1 of c'. We will prove that *the two points A' and A_1 are identical*; but this theorem is not obvious and therefore our demonstration cannot be very simple. The generalization to nonruled surfaces seems to be rather complicated: and we do not occupy ourselves here with such a generalization.

If the point $x=x(u, v)$ generates a ruled surface S, for which $u=$const. and $v=$const. are asymptotic curves, we can suppose (loc. cit., p. 182)

$$x = y + uz \tag{1}$$

in which y and z are functions of v. More clearly, if x_1, x_2, x_3, x_4 are homogeneous projective coordinates of a point of S, we can find eight functions y_i and z_i of v such that

$$x_i = y_i(v) + uz_i(v), \qquad i = 1, 2, 3, 4. \tag{1_{bis}}$$

From the general theory of surfaces, it is known (loc. cit., p. 90) that

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[1] *Geometria Proiettiva Differenziale*, Bologna, Zanichelli.