ERRORS IN HAYASHI'S TABLE OF BESSEL FUNCTIONS FOR COMPLEX ARGUMENTS[1]

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The Project for the Computation of Mathematical Tables is engaged in the computation of an extensive table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for complex arguments.[3] Certain formulas developed for the purpose of checking the tables could also be applied to checking systematically the above-mentioned table of Hayashi (credited to A. Dinnik) and it was deemed worth while to do so.

All values on the rays $\phi=\frac{1}{2}\pi$ and $\phi=\frac{1}{4}\pi$ were checked against the 15-decimal place values computed by us. Those on the ray $\phi=\frac{1}{2}\pi-0.001$ were differenced as a function of r; one error was discovered and corrected.

The entries on all the other rays were differenced as a function of r. In addition, "summation" tests were applied, based on the following theorem due to G. Blanch:

For a fixed r, let $U_0(r, \phi_p)$ represent the real component of $J_0(r, \phi_p)$, where $\phi_p=p\pi/16$, and p ranges from 0 to 7. Similarly, let $V_0(r, \phi_p)$, $U_1(r, \phi_p)$ represent the corresponding values of these functions for a fixed r, and ϕ ranging as before. Then the following formulas hold:

$$\sum_{p=0}^{7} U_0(r, \phi_p) = 8 + \tfrac{1}{2}[J_0(r) - I_0(r)] + R_1, \tag{1}$$

$$\sum_{p=1}^{7} V_0(r, \phi_p) \sin (2\phi_p) = - r^2 + R_2, \tag{2}$$

$$\sum_{p=1}^{7} U_1(r, \phi_p) \cos \phi_p = 2r - \tfrac{1}{2}J_1(r) + R_3, \tag{3}$$

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[1] Published in *Fünfstellige Funktionentafeln*, by K. Hayashi, Berlin, Springer, 1930, pp. 105–109 (from Prof. A. Dinnik, Yekaterinoslaw, 1922). Tabulated in the form $J_0(z)=U_0+iV_0$; $J_1(z)=U_1+iV_1$ for $z=r \exp (i\phi)$. Range of r: 0(0.2)8. Range of ϕ: $0(\pi/16)\pi/2$ for U_0 and V_1; $\pi/16(\pi/16)7\pi/16$ and $\pi/2-0.001$ for V_0 and U_1. To 4 decimal places.

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[3] This work has since been completed.

$$\sum_{p=1}^{7} V_1(r, \phi_p) \sin \phi_p = 2r - \tfrac{1}{2} I_1(r) + R_4, \tag{4}$$

where $|R_i| < 7 \times 10^{-7}$ *for* $r \leqq 8$, $i = 1, 2, 3, 4$.

The summations indicated above were actually performed, and wherever the discrepancy between the theoretical and actual sums exceeded one unit in the fourth decimal place, the source of a possible error was sought, along rays where the differences were irregular. Since each sum contained a group of 7 rounded entries, an occasional discrepancy of two units was to be expected. However, in such cases, the entries giving rise to the discrepancy were checked with special precaution, especially in formulas (2), (3), and (4), where multiplication of the entries by $\sin \phi_p$ or $\cos \phi_p$ might possibly mask an error larger than that shown by the discrepancy.

Because of the great number of errors in the table, it would have been very difficult to localize them by differencing methods alone; but with the aid of the summation tests, the erroneous entries were picked out systematically. The values in error were then re-computed. After all errors were corrected, the differences along the rays were sufficiently regular, and the discrepancies in the summation tests were no larger than 2 units in the fourth decimal place anywhere.

It was obviously impossible to check last-place errors of about a unit, except along the rays which we ourselves have computed. A number of additional errors of one unit were discovered during the recomputation of entries on complementary rays. [Thus, for example, $U_0(r, \phi) = A + B$, where $A = \sum_{k=0}^{\infty} - r^{4k+2} \cos (4k+2)\phi/(k!)^2$, and $B = \sum_{k=0}^{\infty} r^{4k} \cos 4k\phi/(k!)^2$; and $-A + B = U_0(r, \frac{1}{2}\pi - \phi)$. Hence one set of values of A and B yields the entries on two (complementary) rays.] However, it is not deemed worth while to report these errors of one unit. Readers interested in obtaining a list of those which came to light might communicate with the authors. Errors of more than one unit are given below:

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		U_0				V_0	
r	ϕ	*Hayashi's Value*	*Correct Value*	r	ϕ	*Hayashi's Value*	*Correct Value*
6.8	$\pi/16$	0.5338	0.5837	1.0	$\pi/16$	−0.0880	−0.0850
5.2	$2\pi/16$	−0.6625	−0.6601				
7.4	$2\pi/16$	0.5079	2.5079				
1.4	$3\pi/16$	0.7781	0.7731	4.6	$3\pi/16$	0.4125	0.4095
5.6	$3\pi/16$	−2.0436	−2.0636	6.0	$3\pi/16$	4.4654	4.4854

		U_0				V_0	
r	ϕ	*Hayashi's Value*	*Correct Value*	r	ϕ	*Hayashi's Value*	*Correct Value*
5.8	$3\pi/16$	-1.6127	-1.6428				
6.8	$3\pi/16$	2.4803	2.8403				
5.4	$4\pi/16$	-7.6676	-7.6674	6.8	$4\pi/16$	18.0734	18.0736
				7.4	$4\pi/16$	28.6088	27.6088
				8.0	$4\pi/16$	35.0280	35.0167
3.8	$5\pi/16$	-0.9092	-1.0874	0.8	$5\pi/16$	-0.1528	-0.1523
5.0	$5\pi/16$	-8.9930	-9.0929	6.0	$5\pi/16$	-2.8406	-2.8206
6.2	$5\pi/16$	-26.2519	-28.2518				
				6.8	$6\pi/16$	-56.4404	-56.4402
				5.8	$7\pi/16$	-40.0003	-42.8973
5.2	$8\pi/16$	32.5636	32.5836	5.6	$7\pi/16$	-35.0061	-35.0360
6.8	$8\pi/16$	140.1364	140.1362				
8.0	$8\pi/16$	427.5529	427.5641				
				8.0	$\pi/2-0.001$	-3.3587	-3.1989
		U_1				V_1	
0.2	$\pi/16$	0.0966	0.0977	5.6	$\pi/16$	0.0118	0.0818
5.2	$\pi/16$	-0.5248	-0.5255				
				2.2	$2\pi/16$	-0.0745	-0.0754
				8.0	$2\pi/16$	2.6334	2.5000
7.6	$3\pi/16$	-3.8432	-3.8428	5.0	$3\pi/16$	-2.2772	-2.2792
7.8	$3\pi/16$	-2.5846	-2.5841	5.6	$3\pi/16$	-1.7016	-1.7014
8.0	$3\pi/16$	-0.9458	-0.9448	7.4	$3\pi/16$	8.6155	7.3091
				7.8	$3\pi/16$	10.2946	10.2950
				1.0	$4\pi/16$	0.3005	0.3076
5.6	$5\pi/16$	4.6486	4.6489	2.6	$5\pi/16$	0.5418	0.5412
7.4	$5\pi/16$	-44.1809	-42.8744	6.6	$5\pi/16$	-34.4889	-34.4849
				7.6	$5\pi/16$	-52.7084	-52.7088
				7.8	$5\pi/16$	-53.9269	-53.9275
2.6	$6\pi/16$	1.8123	1.8133	8.0	$5\pi/16$	-53.3290	-53.3300
6.2	$6\pi/16$	37.4187	37.4126				
8.0	$6\pi/16$	55.4879	55.3546				
1.4	$7\pi/16$	0.2241	0.2441	0.2	$7\pi/16$	0.0945	0.0985
				2.8	$7\pi/16$	1.7755	2.7755
				5.2	$7\pi/16$	15.7771	15.7764

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