theorem. The conditions under which the inversion is possible are narrow but suitable enough for the purpose at hand. The proof is relegated to the appendix. The term "operational form" is used instead of the more customary word "transform." Although the operational forms of the derivatives of functions with arbitrary initial conditions are derived, very little use is made of them except in the case where the function and all its derivatives vanish when t is zero. The shift operator is neatly applied to obtain Fourier expansions in special cases. A treatment of ordinary differential equations with constant coefficients, Heaviside's unit function, and impulses closes the second part.

The technical problems treated in the third part are too numerous to mention. A feature here is the thorough treatment of several problems with the attainment of numerical results. This is after all the final aim of the technologist. The problems involving partial differential equations are not as clearly stated as those connected with ordinary derivatives.

A final criticism of omission must be made. The central theorem known as the "faltung" or convolution theorem is not even treated as a step-child. This theorem is known to the engineer as Borel's theorem. Under this name it receives only passing attention. The publication of this book is a step in the right direction and the author is to be congratulated for having written a book which will be very valuable to the mathematical technologist.

SAMUEL SASLAW

Théorie Mathématique du Bridge à la Portée de Tous. By Émile Borel and André Chéron. Paris, Gauthier-Villars, 1940. 17+392 pp. Fr. 175.

Here is a detailed study of shuffling, card distributions, the finesse, and other points of the game of bridge by an eminent mathematician and an authority on bridge. A survey of the context naturally enough begins with their initial discussion of the shuffling of the deck.

Shuffling of type A is that generally used in which the deck is separated into two approximately equal parts and then dove-tailed together in small alternating packets. Let a sequence be defined as two adjacent cards in an initial order. If the average number of sequences in the deck broken by one operation of A is S and N is the number of such operations, then NS greater than 150 will result in a well shuffled deck. Another test of the adequacy of shuffling is the fact that the mean number of unicolor sequences (two adjacent cards of the same color) is 12. Shuffling of type B: A packet of a_1 cards is taken off the top of the deck. Then a_2 more are placed on top of the a_1, a_3 more at the bottom, etc. In terms of breaking up sequences, both methods can give satisfactory results, but B more slowly than A.

The authors' conclusion in this first chapter on card shuffling is that elementary shuffling of a deck of cards in the time allotted while another person deals the second deck, produces effectively a random placement of the individual card. It is to be noted that far short of this produces random composition of the four hands, i.e., a random residual of the ordinal position of a card modulo 4. In fact, it is observed by the authors that simply gathering up the tricks at the end of play in a jumbling fashion vitiates the most grave consequences of the absence of shuffling.

They present an interesting method of calculating the probability of a certain side (côté) having a given 26 cards (p. 56) and this is extended (p. 62 ff.) to calculating the probability of the four hands being of specified compositions. The method uses a small table of values constructed for the purpose and the probability of a single combination such as the 4333, 3433, 3343, 3334 distribution at specified positions. This is called the "method of coefficients." Chapters one and two deal with the a priori probabilities and mathematical expectations of the hands and sides. Seventeen tables are included in this part. A sample table is 24 which gives for specified numbers of cards of a suit in a hand the probability that a blank suit, blank suit or singleton, blank suit or singleton or doubleton are present in at least one of the adversaries' hands. The authors point out that some of the tables probably have results counter to the usual notions of what is believed by card players. A case in point is illustrated in Table 25 displaying the percentage frequency of non-occurrence of "accidents"—that is, blank suit or singleton. The accident is the rule for a side (two hands) rather than the exception.

There is a study of the defensive value of hands against bids of the opposition. From a study of the tables on strengths of long-suited hands for offensive and defensive games the conclusion is obtained that long-suited hands are very strong on the offense and very weak on the defense. On the other hand, the possessor of high cards without a long suit is better on the defensive. Chapter 3 also includes many illustrative situations showing how many tricks on an average the first lead (entame) from a suit such as $KQ7$xxx will lose the leading side.

It is emphasized that the abnormal distribution of one suit, for example, a hand with 9 spades, has no effect in giving an expectation of a consequent abnormal distribution of one of the other suits (p.

159). On the other hand, the "law of attraction" between long and short suits and its converse the "law of repulsion" between longs (and shorts) are real aids in, say, locating a given card in two hidden hands. Thus if west has indicated a long suit in diamonds and five spades including a queen are in the two hidden hands, east will have on the average a longer suit in spades than west and consequently the queen of spades more often than west. Exact probabilities of such situations are given in Tables 72 to 101.

Many of the tables of the book are followed with brief summarizing statements which reduce their implications to a working "rule of thumb." For instance the summary of one table (111 bis) reads: "The bidder ought to run a practically unlimited risk (in number of tricks set—with probability of success equal to $\frac{1}{2}$) in order to try to make a bid of game or slam—with the two following exceptions. If it is a question of a game bid redoubled, or of a game bid doubled when the bidder alone is vulnerable, the failure ought never to exceed three tricks." There is considerable discussion of the conditions under which to choose the safe or risky line of play based on the mathematical expectation of the two methods. The question of whether or not to double and redouble is similarly taken up.

The notes comprising the last 128 pages discuss with more refinement some of the questions raised in the book proper. Note 1 considers the theoretical displacements of points in a line segment by random interchange of randomly chosen segmentations of the line. This is then compared to the similar but discontinuous case of shuffling cards. The observation is made that the displacements (i.e., shuffling) are in actuality better than that of the idealized cases studied. For one thing, instead of an independent sequence of "elementary operations"—choosing two random points and interchanging the three segments—in shuffling a deck of cards by the usual method of half the deck in each hand, the two points are replaced by many points. There is a consequent diminishing of the number of sub-segments (cards) not displaced, for the same total number of division points used.

Note 2 goes into a detailed study of the method used in enumerating the permutations of a given set of four partitions of 13. Thus the four hands might be $AAAB$ where A is the hand (3334) and B is (4441). From these values are obtained the total probability of a given designated deal. Perhaps the most interesting idea to the reader familiar with probability concepts is that of "virtual" probability discussed in Note 3. For the case of dependent probabilities the product rule holds for virtual probabilities of the separate events, as it holds for real probabilities of independent events. The specific situation is: Consider N objects of K types so that $N=A_1+A_2+\cdots+A_K$. Choose by chance n of them. The probability that the composition of n is $a_1, a_2, \cdots, a_K$ of the K types is given by

$$P_{a_1, a_2, \cdots, a_K} = p_1 p_2 \cdots p_K$$

where

$$p_i = C_{A_i}^{a_i}/(C_N^n)^{A_i/N}.$$

p_i is the "virtual" probability that a_i of the A_i objects will be selected. This idea is applied to the selection of bridge hands and is the means of easily calculating through tables (114, 115) the real probability of one or more hands being of specified composition.

Note 4 gives a short discussion on the theory of finesse and on the "squeeze" play, with the observation that actual play generally amounts to choices between the advantages and disadvantages of the finesse and the squeeze play for the hands under consideration. A lengthy discussion with illustrations on the variation of probabilities in the course of play is the province of Note 6. The new probabilities are of course obtained by the use of Bayes' theorem. From the bidding and one or two rounds of play, certain restrictions can be deduced on the distribution of residue of a given suit. For purposes of finesse, the probabilities pertaining to positions of certain high cards must then be revalued as just indicated.

One of the most interesting tables is 129 which lists in order of decreasing frequency the distribution possibilities of the two hidden hands after the dummy has been displayed. The order is arrived at by a method explained in Note 7 as the "complement to the method of coefficients" mentioned previously. Some common errors of reasoning due mostly to misapplications or non-applications of Bayes' formula are elucidated in Note 8. The use of "psychological" probabilities for complete use of the formula is also given here. The final note of the book deals at length with a problem of finesse which again involves psychological probabilities, i.e., how the individual player reacts to given situations (e.g., can he be depended upon to cover the queen with the king of the suit, or to play the lowest card when following suit with a weak card), as well as information obtained from the cards already played.

It may be said that the book is written with the logically minded rather than the mathematically equipped person continually in view. The 134 tables involve a prodigious amount of computation. The fact that the calculating machine with automatic retransmission of the product for a second multiplication was used instead of factorial and logarithm approximations, and that all probabilities were computed in at least two different ways, leads the authors to make highly confident statements as to the accuracy of the figures.

The reviewer's reaction to a first reading of this work was mainly a sense of increased appreciation of the fundamentals and potentialities of the game—much as one gets from a study of a good chess book. Further study of some of the vital points would undoubtedly save many a trick and generally improve the game of most of us.

JOSEPH A. GREENWOOD

Introduzione al Pensiero Matematico. By Friedrich Waismann. Translated by Ludovico Geymonat. (Biblioteca di Cultura Scientifica, vol. 111.) Torino, Einaudi, 1939. 325 pp. Lire 20.

This book, originally published in German (Vienna, 1936) and now translated into Italian, is a popular exposition of the fundamental concepts and views underlying modern mathematics. Special attention is paid to those sides of present day mathematics which may seem to involve "philosophical" problems, and a manifold of philosophical suggestions are advanced, mostly credited to an unpublished manuscript by Wittgenstein. The expository part of the book combines, to a rare degree, accuracy and comprehensibility.

After having traced in outline the historical growth of our present numbers system, the author envisages the question, often raised by philosophers: "How can the postulation of 'new' numbers be justified? Do there 'exist,' e.g., any irrational and complex numbers?" It is shown that these questions are equivalent to the questions whether the enlarged mathematical calculus is consistent, and whether it can be given an interpretation. The answer by a reference to geometrical facts is rejected as unsatisfactory, since proof of the consistency of geometry will depend on the assumption that our number system is consistent. The problem of the existence of the non-natural numbers—as far as it can be separated from the consistency problem—is now solved by giving the well known construction of them on the basis of the natural numbers. Following Skolem, the author then shows how elementary arithmetic can be strictly developed on the minimum basis of three undefined notions (natural number, successor, identity) and one inductive definition (of addition) by using the method of complete induction. The present situation of the "Grundlagenforschung" is reviewed, mention being made of theorems by Gödel and Skolem.

The "philosophical" ideas, expressed in the book, are unfortunately