NOTE ON THE COMPUTATION OF THE DIFFERENCES OF THE Si(x), Ci(x), Ei(x) AND $-$Ei($-x$) FUNCTIONS

MILTON ABRAMOWITZ

In a recent article,[1] Lowan reached the conclusion that the second differences of the functions Si(x), Ci(x) and Ei(x) for values of x from 0.1 to 2 at intervals of 0.0001 may be approximated to 12 decimal places by certain expressions in closed form, which may be computed with the aid of tabulated functions.

The above functions and also $-\mathrm{Ei}(-x)$ were computed to 12 decimal places by adding an appropriate number of terms in their power series expansion.[2] In order to check the accuracy of the computations, the second differences of the computed values were required to agree to 12 decimal places with the values of the approximate expressions mentioned above.

The functions Ci(x), Ei(x), $-\mathrm{Ei}(-x)$ have a logarithmic singularity at the origin. For this reason Lowan's conclusion could not be extended to the interval $0<x<0.1$.

If $\phi(x)$ represents any of the functions $\mathrm{Ci}(x)-\log_e x$, $\mathrm{Ei}(x)-\log_e x$, $-\mathrm{Ei}(-x)+\log_e x$, it will be shown that the second difference of $\phi(x)$ may be approximated by $h^2\phi''(x)$ to 12 decimal places. Let

$$R(x) = [\phi(x+h) - 2\phi(x) + \phi(x-h)] - h^2\phi''(x).$$

Substituting for $\phi(x+h)$ and $\phi(x-h)$ their Taylor expansions, we get

$$R(x) = 2\left[\frac{h^4}{4!}\phi^4(x) + \frac{h^6}{6!}\phi^6(x) + \cdots\right]$$

whence

$$\text{(A)} \qquad |R(x)| < 2\left[\frac{h^4}{4!}\{\phi^4(x)\} + \frac{h^6}{6!}\{\phi^6(x)\} + \cdots\right]$$

where $\{\phi^{2k}(x)\}$ is an upper bound of the modulus of the $2k$th derivative.

[1] A. N. Lowan, *On the computation of second differences of the* Si(x), Ei(x), *and* Ci(x) *functions*, this Bulletin, vol. 45 (1939), pp. 583–588. This will be referred to as A.N.L.

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Case of the function $\mathrm{Ei}(x)-\log_e x$. From

$$\frac{d}{dx}(\mathrm{Ei}\,(x)-\log_e x)=\frac{e^x-1}{x}=\int_0^\infty e^{x(1-t)}dt-\int_0^\infty e^{xt}dt$$

(see A.N.L.) we get

$$\frac{d^{2k}}{dx^{2k}}(\mathrm{Ei}\,(x)-\log_e x)=\int_0^\infty (1-t)^{2k-1}e^{x(1-t)}dt-\int_0^\infty t^{2k-1}e^{xt}dt,$$

and, in view of (9) and (10) of A.N.L.,

$$\left|\frac{d^{2k}}{dx^{2k}}(\mathrm{Ei}\,(x)-\log_e x)\right|<\frac{e^x}{2k}.$$

Making use of this result (A) yields

$$|\,R(x)\,|<2e^x\left\{\frac{h^4}{4!}\,\frac{1}{4}+\frac{h^6}{6!}\,\frac{1}{6}+\cdots\right\}\sim\frac{e^x}{2}\,\frac{h^4}{4!}\,\frac{1}{1-h}.$$

For $h=10^{-4}$ and $0<x<0.1$, the last expression is smaller than 10^{-17}. We thus reach the conclusion that for these values of h and x, the second differences of $\mathrm{Ei}(x)-\log_e x$ may be approximated by

$$h^2\frac{d^2}{dx^2}(\mathrm{Ei}\,(x)-\log_e x).$$

From the series expansion for the function $\mathrm{Ei}(x)-\log_e x$ it is easily seen that the second difference of this function may be approximated to 12 decimal places by

$$h^2\left(\frac{1}{2}+\frac{x}{3}+\frac{1}{4}\,\frac{x^2}{2!}+\frac{1}{5}\,\frac{x^3}{3!}\right).$$

In an entirely similar manner it may be shown that $1/(2k)$ is an upper bound of the derivatives of the functions $-\mathrm{Ei}(-x)+\log_e x$ and $\mathrm{Ci}(x)-\log_e x$, and if $h=10^{-4}$ and $0<x<0.1$, the second difference of the latter functions may be approximated to 12 decimal places by the respective expressions

$$h^2\left(-\frac{1}{2}+\frac{x}{3}-\frac{1}{4}\,\frac{x^2}{2!}+\frac{1}{5}\,\frac{x^3}{3!}\right),\qquad h^2\left(-\frac{1}{2}+\frac{1}{4}\,\frac{x^2}{2!}\right).$$

New York City