GROUPS OF MOTIONS IN CONFORMALLY FLAT SPACES. II

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1. **Introduction.** In a previous paper with a similar title,* we have shown that all groups of motions admitted by a conformally flat metric space V_n must be subgroups of the general conformal group G_N of $N=\frac{1}{2}(n+1)(n+2)$ parameters generated by†

$$\xi^i = b^i + a_0 x^i + x^i a_j x^j - \tfrac{1}{2} a_i e_i e_j (x^j)^2 + b_j^i x^j, \qquad e_i = \pm 1. \tag{1}$$

In (1), the b_j^i satisfy the relations $e_i b_j^i + e_j b_i^j = 0$, ($i, j$ not summed). Otherwise the a's and b's in (1) are arbitrary.

To define a group of motions of V_n, the ξ^i must satisfy the equations

$$\xi^k \frac{\partial h}{\partial x^k} + h \frac{\partial \xi^i}{\partial x^i} = 0, \qquad i \text{ not summed}, \tag{2}$$

and the coordinates x^i of (2) are such that $g_{ij} = e_i \delta_j^i h^2$. Hence in this coordinate system, the metric has the form

$$ds^2 = h^2 \sum e_i (dx^i)^2. \tag{3}$$

In this paper we shall consider the simplest subgroups of G_N, and determine the nature of the function h corresponding to each. Also we give a restatement of Theorem 2 of I, since it is not complete as given.

2. **The group** G_N. The basis of the group G_N may be taken in the form

$$P_i = p_i, \tag{4}$$

$$S_{ij} = e_i x^i p_j - e_j x^j p_i, \qquad i, j \text{ not summed}, \tag{5}$$

$$U = x^i p_i, \tag{6}$$

$$V_i = 2x^i x^j p_j - e_i e_j (x^j)^2 p_i, \tag{7}$$

where $p_i = \partial / \partial x^i$; and its commutators are‡

* *Groups of motions in conformally flat spaces*, this Bulletin, vol. 42 (1936), pp. 418–422. The results of this paper (which we refer to as I) will be assumed known.

† All small Latin indices take the values 1, 2, $\cdots$, n, with $n>2$, unless otherwise noted.

‡ S. Lie, *Theorie der Transformationsgruppen*, vol. 3, pp. 321–334.

(8a) $(P_i, P_j) = 0,$

(8b) $(P_i, U) = P_i,$

(8c) $(P_i, S_{jk}) = e_i\delta_{ij}P_k - e_k\delta_{ik}P_j,$

(8d) $(P_i, V_j) = 2\delta_{ij}U - 2e_jS_{ij},$

(8e) $(S_{ij}, S_{kl}) = e_j\delta_{jk}S_{il} - e_j\delta_{jl}S_{ik} - e_i\delta_{ik}S_{jl} + e_i\delta_{il}S_{jk},$

(8f) $(S_{ij}, U) = 0,$

(8g) $(S_{ij}, V_k) = e_i\delta_{jk}V_i - e_j\delta_{ik}V_j,$

(8h) $(U, V_i) = V_i,$

(8i) $(V_i, V_j) = 0.$

The four types of symbols, P_i, S_{ij}, U, V_i, will be considered singly and in various combinations to form the subgroups to be discussed.

3. **Subgroups of one type of symbol.** We consider first the subgroups with symbols*

(a) $[P_\alpha]$, (b) $[U]$, (c) $[S_{\alpha\beta}]$, (d) $[V_\alpha]$.

The notation $[P_\alpha]$ means $[P_1, P_2, \cdots, P_r]$, and similarly for other expressions of this nature. That each of (a)–(d) forms a subgroup follows from (8a), (8e), (8i).

For (a), we have from (4), $\xi_\alpha^i = \delta_\alpha^i$, and (2), written in the form

$$\xi_\alpha^k \frac{\partial h}{\partial x^k} + h \frac{\partial \xi_\alpha^i}{\partial x^i} = 0, \qquad i \text{ not summed},$$

becomes

$$\frac{\partial h}{\partial x^\alpha} = 0. \tag{9}$$

Hence (a): $h = h(x^{r+1}, \cdots, x^n)$. In case $r = n$, h is constant, and the V_n is flat.

The finite equations of the group $[P_\alpha]$ are

$$x'^i = x^i + a^\alpha\delta_\alpha^i \tag{10}$$

with parameters a^α. Because of the form of (10), we call this group the T_r of *translations*. However, the group of motions $[P_\alpha]$ is not a group of translations of the V_n unless h = constant,† that is, unless V_n is flat.

* Greek letters take the values $1, 2, \cdots, r$, with $r \leqq n$.

† L. P. Eisenhart, *Continuous Groups of Transformations*, p. 212. We refer to this book as CG.

For (b), we have $\xi^i = x^i$, and (2) becomes

$$(11)\qquad x^i \frac{\partial h}{\partial x^i} = -h.$$

Hence h is homogeneous of degree -1, that is,

$$(b)\qquad h = \frac{1}{x^1}\phi\left(\frac{x^2}{x^1}, \cdots, \frac{x^n}{x^1}\right),$$

say, where ϕ is an arbitrary function of its arguments.

The finite equations of the group $[U]$ are $x'^i = ax^i$, the group of *dilations*.

For (c), we find

$$\xi^i_{\alpha\beta} = e_\alpha \delta^i_\beta x^\alpha - e_\beta \delta^i_\alpha x^\beta, \qquad \alpha \neq \beta,$$

as the vector components of the group $[S_{\alpha\beta}]$ of $\frac{1}{2}r(r-1)$ parameters. The equations (2) which must be satisfied for each $\xi^i_{\alpha\beta}$ now become

$$(12)\qquad X_{\alpha\beta}h \equiv e_\alpha x^\alpha \frac{\partial h}{\partial x^\beta} - e_\beta x^\beta \frac{\partial h}{\partial x^\alpha} = 0, \quad \alpha, \beta \text{ not summed}.$$

These equations have as general solution,

$$(c)\qquad h = h(u; x^{r+1}, \cdots, x^n),$$

where $u = \sum e_\alpha (x^\alpha)^2$.

In obtaining this, we use the fact that the system (12) contains $r-1$ independent equations, since

$$e_\alpha x^\alpha X_{\beta\gamma} + e_\beta x^\beta X_{\gamma\alpha} + e_\gamma x^\gamma X_{\alpha\beta} \equiv 0, \qquad \text{no summing},$$

and it is also a complete system.*

The group $[S_{\alpha\beta}]$ has the finite equations

$$x'^\alpha = a^\alpha_\beta x^\beta, \qquad x'^A = x^A, \qquad A = r+1, \cdots, n,$$

with

$$\sum e_\alpha a^\alpha_\beta a^\alpha_\gamma = e_\beta \delta_{\beta\gamma}.$$

We call this group of $\frac{1}{2}r(r-1)$ parameters, the $R_{r(r-1)/2}$ of *rotations*.†

The vector components for the group (d) are

$$\xi^i_\alpha = 2x^i x^\alpha - e_\alpha \delta^i_\alpha R,$$

* Goursat, *Mathematical Analysis*, vol. 2, part 2, p. 270.

† CG, p. 57, problem 12.

where $R=\sum e_i(x^i)^2$. Equations (2) reduce, for this case, to

$$2x^\alpha x^i \frac{\partial h}{\partial x^i} - e_\alpha R \frac{\partial h}{\partial x^\alpha} + 2hx^\alpha = 0. \tag{13}$$

If we put $\lambda = x^i \partial h/\partial x^i$, (13) may be written in the form

$$\frac{2(\lambda + h)}{R} = \frac{e_\alpha}{x^\alpha} \frac{\partial h}{\partial x^\alpha}, \qquad \alpha \text{ not summed.}$$

Since the left member of this equation is independent of α, we may write

$$\frac{e_\alpha}{x^\alpha} \frac{\partial h}{\partial x^\alpha} = \frac{e_\beta}{x^\beta} \frac{\partial h}{\partial x^\beta},$$

which simplifies to (12), and hence h is of the form for (c). Using this form for h in (13), we obtain on reduction,

$$(u - v) \frac{\partial h}{\partial u} + \sum x^A \frac{\partial h}{\partial x^A} = - h, \qquad A = r + 1, \cdots, n, \tag{14}$$

with

$$v = \sum e_A (x^A)^2.$$

The equation (14) has as solution

$$h = \frac{1}{R} \phi\left(\frac{x^{r+1}}{R}, \cdots, \frac{x^n}{R}\right).$$

In case $r=n$, $h=a/R$, with a constant, and the V_n is flat.*

The finite equations for the group $[V_\alpha]$ are†

$$x'^i = \frac{x^i - \frac{1}{2} R \delta_\alpha e_\alpha a_\alpha}{1 - a_\alpha x^\alpha + \frac{1}{4} e_\alpha e_\beta a_\alpha^2 (x^\beta)^2}.$$

4. **Subgroups with two types of symbols.** We consider in this section the simplest subgroups with two types of symbols. These are:

(e) $[P_\alpha, S_{\beta\gamma}]$, (f) $[P_\alpha, U]$, (g) $[S_{\alpha\beta}, U]$,

(h) $[V_\alpha, U]$, (i) $[S_{\alpha\beta}, V_\gamma]$.

Each of these we discuss briefly.

(e). The function h has the same form as for (a) since equations (12) are satisfied identically if (9) are.

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* L. P. Eisenhart, *Riemannian Geometry*, p. 85.

† Lie, loc. cit., p. 350.

(f). Using the form of h for (a) in (11), we see that h is homogeneous of degree -1 in $x^{r+1}, \cdots, x^n$, that is, we may write

$$h = \frac{1}{x^{r+1}} \phi\left(\frac{x^{r+2}}{x^{r+1}}, \cdots, \frac{x^n}{x^{r+1}}\right).$$

If $r=n$, there is no solution.

(g). If we substitute for h in (11) its value as determined from (c), we obtain

$$2u\frac{\partial h}{\partial u} + x^A \frac{\partial h}{\partial x^A} = -h, \qquad A = r+1, \cdots, n. \tag{15}$$

Hence,

$$h = \frac{1}{u^{1/2}} \phi\left(\frac{x^{r+1}}{u^{1/2}}, \cdots, \frac{x^n}{u^{1/2}}\right).$$

(h). Equations (11) and (13) show $\partial h/\partial x^\alpha = 0$, so that h is the same as in (f). If $r=n$, there is no solution.

(i). For (d), we have seen that (13) imply (11), that is, the form of h for (i) is the same as that for (d).

5. **Subgroups with three and four types of symbols.** Of the four possibilities $[P_\alpha, S_{\beta\gamma}, V_\delta]$, $[P_\alpha, S_{\beta\gamma}, U]$, $[P_\alpha, V_\beta, U]$, $[S_{\alpha\beta}, V_\gamma, U]$, only the second and fourth give subgroups:

(j) $[P_\alpha, S_{\beta\gamma}, U]$, (k) $[S_{\alpha\beta}, V_\gamma, U]$.

For (j), the P_α, $S_{\beta\gamma}$ imply $h=h(x^{r+1}, \cdots, x^n)$, and then U shows h is the same form as in (f). There is no solution of $r=n$.

The form of h for (k) will be the same for (h), as follows from (i), that is, h will have the same form as for (f). If $r=n$, there is no solution.

The simplest four type symbol subgroup is

(l) $[P_\alpha, V_\beta, S_{\gamma\delta}, U]$.

It is easily seen that the solution for h is the same as for (f), and there is no solution for $r=n$.

6. **Indices in different ranges.** So far, we have considered only subgroups whose symbol indices all have the same range, $1, \cdots, r$. In this section we discuss cases (e), (i), (j), (k), and (l) with the indices for the various types of symbols in different ranges.

Case (m): $[P_i, S_{jk}]$. Let i range through $1, \cdots, r$, and j, k through any set of t indices, $s_1, s_2, \cdots, s_t$, with $s_1 < s_2 < \cdots < s_t$. Then either:

(m_1) $s_t \leqq r$, (m_2) $s_1 \leqq r,\ s_t > r$, (m_3) $s_1 > r$.

For case (m_1), equations (9) imply (12) with α, β in the range $s_1, s_2, \cdots, s_t$. Hence h has the same form as in (a).

In the second case, (m_2), there must be a common index in $(1, \cdots, r)$ and $(s_1, \cdots, s_t)$, say β. Then, in (8c), choose $i=j=\beta$, and $k=s_t$. This gives

$$(P_\beta, S_{\beta s'}) = e_\beta P_{s'}, \qquad s' = s_t,$$

which is not in the set $[P_\alpha]$. Hence, this case is impossible.

For case (m_3), the two sets of indices have no index in common, and we must have $t \geqq 2$. Without loss of generality, we may take the set $s_1, \cdots, s_t$ to be $r+1, r+2, \cdots, r+t$. The form of h is easily seen to be

$$h = h(v_t;\ x^{r+t+1}, \cdots, x^n), \quad v_t = \sum_{r+1}^{r+t} e_J (x^J)^2.$$

Case (n): $[S_{jk}, V_i]$. As in case (m), there are three possibilities, only the first and third being possible. If we let i take the range $1, \cdots, r$, then if $s_t \leqq r$, h has the same form as for (d). If $s_1 > r$, we may let j, k have the range $r+1, \cdots, r+t$. Then h must satisfy (13), and (12) with the indices in this latter range. Since (13) implies (12), we must have $h = h(u;\ v_t;\ x^{r+t+1}, \cdots, x^n)$. Using this form for h in (13), we obtain

$$(u - w)\frac{\partial h}{\partial u} + v_t \frac{\partial h}{\partial v_t} + x^B \frac{\partial h}{\partial x^B} = -h, \qquad B = r + t + 1, \cdots, n,$$

with $w = \sum e_B (x^B)^2$. This equation has as solution

$$h = \frac{1}{R - v_t}\,\phi\left(\frac{v_t}{R - v_t};\ \frac{x^{r+t+1}}{R - v_t}, \cdots, \frac{x^n}{R - v_t}\right).$$

With three types of symbols, we consider first $[P_i, S_{jk}, U]$, and let $i = 1, \cdots, r$. If the indices of S_{jk} are all contained in the range $1, \cdots, r$, h has the same form as for $[P_\alpha, U]$. Otherwise, we must have all j, k indices outside the range $1, \cdots, r$. Then we have: (o) $[P_a, S_{JK}, U]$, and $h = h(v_t;\ x^B)$, using the notation of case (n). With this value of h in (11) we obtain equation (15) with u replaced by v_t. Hence,

$$h = \frac{1}{v_t^{1/2}}\,\phi\left(\frac{x^{r+t+1}}{v_t^{1/2}}, \cdots, \frac{x^n}{v_t^{1/2}}\right).$$

As the next case we consider $[V_\alpha, S_{jk}, U]$. If the j, k indices are included in $1, \cdots, r$, we get the same form for h as in $[V_\alpha, U]$. If not we must have j, k in the range J, K, to give: (p) $[V_\alpha, S_{JK}, U]$. The symbols V_α, U imply $h=h(x^{r+1}, \cdots, x^n)$, and then the symbols S_{JK} imply $h=h(v_t; x^B)$, the same as in (o).

The other two possibilities $[P_i, S_{jk}, V_l]$, $[P_i, V_j, U]$ are easily shown to be impossible, no matter in what ranges we choose the indices of the various symbols.

For four types we have $[P_\alpha, S_{jk}, V_l, U]$. If j, k are in the J, K range, we have a contradiction from (P_α, V_l), no matter what range l has. The only other choice is j, k included in the $1, \cdots, r$ range. Then, from (P_α, V_l), we must have l in this range also. This gives

(q) $[V_\alpha, S_{\alpha'\beta'}, V_{\gamma'}, U]$, α', β', γ' range included in $1, \cdots, r$,

and h has the same form as for (f), as easily follows.

7. **Summary.** We give here a summary of the various forms for h corresponding to the subgroups considered.

(a) $[P_\alpha]$, $h = h(x^{r+1}, \cdots, x^n)$;

(b) $[U]$, $h = \dfrac{1}{x^1}\,\phi\left(\dfrac{x^2}{x^1}, \cdots, \dfrac{x^n}{x^1}\right)$;

(c) $[S_{\alpha\beta}]$, $h = h(u; x^{r+1}, \cdots, x^n)$;

(d) $[V_\alpha]$, $h = \dfrac{1}{R}\,\phi\left(\dfrac{x^{r+1}}{R}, \cdots, \dfrac{x^n}{R}\right)$;

(f) $[P_\alpha, U]$, $h = \dfrac{1}{x^{r+1}}\,\phi\left(\dfrac{x^{r+2}}{x^{r+1}}, \cdots, \dfrac{x^n}{x^{r+1}}\right)$, $r = n$, no solution;

(g) $[S_{\alpha\beta}, U]$, $h = \dfrac{1}{u^{1/2}}\,\phi\left(\dfrac{x^{r+1}}{u^{1/2}}, \cdots, \dfrac{x^n}{u^{1/2}}\right)$;

(m_3) $[P_\alpha, S_{IJ}]$, $h = h(u; x^B)$;

(n_3) $[V_\alpha, S_{IJ}]$, $h = \dfrac{1}{R - v_t}\,\phi\left(\dfrac{v_t}{R - v_t}; \dfrac{x^B}{R - v_t}\right)$;

(o_3) $[P_\alpha, S_{IJ}, U]$, $h = \dfrac{1}{v_t^{1/2}}\,\phi\left(\dfrac{x^B}{v_t^{1/2}}\right)$;

(e) $[P_\alpha, S_{\beta\gamma}]$, and ($\mathrm{m}_1$) $[P_\alpha, S_{\beta'\gamma'}]$, h as in (a);

(i) $[S_{\alpha\beta}, V_\gamma]$, and ($\mathrm{n}_1$) $[V_\alpha, S_{\beta'\gamma'}]$, h as in (d);

(h) $[V_\alpha, U]$, (j) $[P_\alpha, S_{\beta\gamma}, U]$, (k) $[S_{\alpha\beta}, V_\gamma, U]$,

(l) $[P_\alpha, V_\beta, S_{\gamma\delta}, U]$, ($\mathrm{o}_1$) $[P_\alpha, S_{\beta'\gamma'}, U]$,

(p_1) $[V_\alpha, S_{\beta'\gamma'}, U]$, (q) $[V_\alpha, S_{\beta'\gamma'}, V_{\delta'}, U]$, all have h as in (f);

(p$_3$) $[V_\alpha, S_{IJ}, U]$, h as in (o$_3$).

In the above summary we have used the following notation:

$$R = \sum e_i(x^i)^2, \qquad u = \sum e_\alpha(x^\alpha)^2, \qquad v_t = \sum e_I(x^I)^2,$$

$i=1, \cdots, n$; Greek letters have the range $1, \cdots, r$; $I, J=r+1, \cdots, r+t$; $A=r+1, \cdots, n$; primed Greek letters have a range contained within $1, \cdots, r$; $B=r+t+1, \cdots, n$.

8. **Restatement of Theorem** 2 **of** I. In the proof of this theorem, the possibility $a_0=b^i=a_i=0$ was omitted. In this case, ξ^i has the form $\xi^i=b_j^i x^j$, and the function $f(R)$ is arbitrary. The group for this case is evidently the rotation group $[S_{ij}]$ of $\frac{1}{2}n(n-1)$ parameters. It is not difficult to show that the subgroups corresponding to the two cases mentioned in the theorem are $[ce_iP_i+V_i, S_{jk}]$ for $f(R)=(\alpha R+\beta)^2$ and $[S_{ij}, U]$ for $f(R)=\alpha R$. We may thus state the corrected theorem in the form:

THEOREM. *Every metric space with quadratic form* $\sum e_i(dx^i)^2/f(R)$ *admits the rotation group* $[S_{ij}]$ *as a group of motions. The only metric spaces with this quadratic form which admit other groups of motions are spaces of constant curvature, and f has the form* $f(R)=(\alpha R+\beta)^2$, *and the group is* $[ce_iP_i+V_i, S_{jk}]$, *and spaces with* $f(R)=\alpha R$, *in which case the group is* $[S_{ij}, U]$.

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