It remains an unsolved problem whether or not generalizations of the theorems of this paper can be established which would include the border and frontier operators or which would not require that the space be dense in itself.

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A NOTE ON A PAPER BY J. A. TODD*

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In a recent paper† entitled *A note on the linear fractional group*, Todd obtained an abstract definition for the group $LF(2, 2^n)$ in terms of $n+2$ generators. Apparently he gave no consideration to the question of the independence of the defining relations, for they can be considerably simplified. First, in view of the condition $RS_i = S_{i+1}R$ (which is the same as $R^iS_0R^{-i} = S_i$), the three generators U, R and S_0 are sufficient to generate the entire group. If we give a definition in terms of these three generators alone, the relations

$$S_i^2 = 1, \qquad i \neq 0, \qquad RS_i = S_{i+1}R$$

may be discarded, and any S_i $(i \neq 0)$ which appears in the remaining conditions may be replaced by its definition in terms of R and S_0. Next, the $C_{n-1,2}$ conditions $S_iS_j = S_jS_i$ can be replaced by the $n-1$ conditions

$$S_0S_i = S_iS_0, \qquad i = 1, 2, \cdots, n-1.$$

For suppose $j - i = \alpha$. Then, from $S_0S_\alpha = S_\alpha S_0$, we get

$$R^i(S_0S_\alpha)R^{-i} = R^i(S_\alpha S_0)R^{-i}, \qquad S_iS_j = S_jS_i.$$

Writing $S_0S_i = S_iS_0$ in terms of R and S_0 only

$$S_0R^iS_0R^{-i} = R^iS_0R^{-i}S_0, \qquad (S_0R^iS_0R^{-i})^2 = 1.$$

Thus, for the three generators U, R and S_0, we require only $n+5$ conditions

$$R^{2^n-1} = U^3 = (UR)^2 = (US_0)^2 = S_0^2 = 1, \ (S_0R^iS_0R^{-i})^2 = 1,$$

$$R^nS_0R^{-n+1} = S_0^{a_0}RS_0^{a_1}RS_0^{a_2} \cdots RS_0^{a_{n-1}}R^{-n+2}.$$

But even these three generators are not independent. For the relation $(UR)^2 = 1$ permits us to consider U and R as being equivalent to

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† Journal of the London Mathematical Society, vol. 2 (1936), pp. 103–107.

two generators of periods two and three, and these two generators can be shown to give rise to the entire group.* To put it differently, S_0 is expressible in terms of U and R. The question now arises whether the above $n+5$ conditions, when expressed in terms of U and R alone, might not involve redundancies. For example, $LF(2, 2^3)$ can be completely defined by means of only four conditions on two generators of periods two and three.† However, a general study of the independence of the relations would require the expression for S_0 in terms of U and R; this the author has been unable to obtain. It most likely would be found to involve the coefficients in the equation satisfied by the primitive root ϵ.

It nevertheless seemed an interesting problem to investigate one or two further special cases in order to see whether in those particular cases the number of defining relations could not be reduced.

Consider first $LF(2, 2^4)$. On referring to the Bulletin paper, it is seen that Todd's generators would satisfy (2, 3, 15; 17). (This is the only one of the three possibilities for which $p=15$.) Hence, the operators P and Q introduced in that paper would satisfy

$$P^{15} = (QP^2)^3 = (QP^3)^2 = (Q^9P)^2 = 1.$$

By making use of the coset enumeration process devised by Todd and Coxeter,‡ the author has been able to show that it is sufficient to add only one further relation to the above set, namely, $(Q^2P^8)^2=1$, to have a complete definition of $LF(2, 2^4)$. The basic subgroup used in the procedure was $\{Q, PQP^2\}$, of order 34, so that it was necessary to enumerate only 120 cosets. By making use of the substitution

$$P^2 = \overline{Q}, \quad Q = \overline{P}^2, \quad \overline{P}^{17} = 1,$$

it follows at once§ that a second complete definition is given by

$$P^{17} = (QP^2)^3 = (QP^3)^2 = (Q^8P)^2 = (Q^4P^4)^2 = 1.$$

In view of the fact that $LF(2, 2^4)$ has only three abstract definitions in terms of two generators of periods two and three, it seemed worth while to investigate the one remaining definition. Using the enumera-

* A. Sinkov, *On generating the simple group $LF(2, 2^N)$ by two operators of periods two and three*, this Bulletin, vol. 44 (1938), p. 455. This paper will hereafter be referred to as the Bulletin paper.

† A. Sinkov, *Necessary and sufficient conditions for generating certain simple groups by two operators of periods two and three*, American Journal of Mathematics, vol. 59 (1937), p. 70.

‡ Proceedings of the Edinburgh Mathematical Society, (2), vol. 5 (1937), pp. 26–34.

§ Compare the Bulletin paper, p. 454.

tion process once again, and with the same basic subgroup, it turned out that a complete definition is given in this instance by

$$P^{17} = (QP^2)^3 = (QP^3)^2 = (Q^9P)^2 = (Q^2P^{13})^2 = 1.$$

This now permits us to state the following necessary and sufficient condition:

THEOREM. *In order for two operators S and T of periods three and two, respectively, to generate $LF(2, 2^4)$, it is necessary and sufficient that they satisfy one of the following sets of relations*:

$$P^{15} = (QP^2)^3 = (QP^3)^2 = (Q^9P)^2 = (Q^2P^8)^2 = 1, \tag{1}$$

$$P^{17} = (QP^2)^3 = (QP^3)^2 = (Q^8P)^2 = (Q^4P^4)^2 = 1, \tag{2}$$

$$P^{17} = (QP^2)^3 = (QP^3)^2 = (Q^9P)^2 = (Q^2P^{13})^2 = 1, \tag{3}$$

where $P = (ST)^{-1}$, $Q = (ST)^2S$.

If we make use of the group $G^{m,n,p}$ defined by Coxeter,* the above theorem takes a much more elegant form, since the three sets of relations reduce to two:

$$G^{3,15,17}, \quad (C^4B^8)^2 = 1, \tag{4}$$

$$G^{3,17,17}, \quad (C^4B^{13})^2 = 1. \tag{5}$$

In the case of $LF(2, 2^5)$, the basic subgroup $\{Q, PQP^2\}$, of order 62, required the enumeration of 528 cosets. Todd's generators satisfy the relations (2, 3, 11; 31) and a complete definition was obtained in the form

$$P^{11} = (QP^2)^3 = (QP^3)^2 = (Q^{16}P)^2 = (Q^2P^8QP^{10})^2 = (Q^2P^{10}QP^8)^2 = 1.$$

As before, this leads to a second complete definition

$$P^{31} = (QP^2)^3 = (QP^3)^2 = (Q^6P)^2 = (Q^4P^2Q^5P^4)^2 = (Q^4P^4Q^5P^2)^2 = 1,$$

both of which are equivalent to the single definition

$$G^{3,11,31}, \quad (C^4B^8C^2B^{10})^2 = (C^4B^{10}C^2B^8)^2 = 1.$$

We see then that when n is 3, 4 or 5, $n+1$ conditions are sufficient to yield a complete definition of $LF(2, 2^n)$. This would lead to the conjecture that the $n+5$ relations deducible from Todd's definition are, in general, not all independent.

A similar sort of treatment is possible in the case of Todd's definition for $LF(2, p^n)$. Here it follows from the definition $RS_i = S_{i+2}R$ that

* *The abstract groups $G^{m,n,p}$*, Transactions of this Society, vol. 45 (1939), pp. 73–150.

$S_2, S_3, \cdots, S_{n-1}$ are expressible in terms of either R and S_0 or R and S_1 according as the subscript is even or odd. Hence U, R, S_0, and S_1 will yield a complete definition in the case $p^n \equiv -1 \pmod 4$ if they satisfy the $2n+5$ conditions

$$R^{(p^n-1)/2} = U^3 = (UR)^2 = (US_0)^2 = S_0^p = S_1^p = 1,$$

$$S_0S_i = S_iS_0; S_1S_j = S_jS_1, \; i = 1, 2, \cdots, n-1; j = 2, 3, \cdots, n-1,$$

$$RS_{n-2} = S_0^{a_0}S_1^{a_1} \cdots S_{n-1}^{a_{n-1}}R, \qquad RS_{n-1} = S_0^{b_0}S_1^{b_1} \cdots S_{n-1}^{b_{n-1}}R.$$

In these relations the S_i $(i>1)$ are to be replaced by their expressions in terms of R and S_0 or R and S_1. Should $p^n \equiv +1 \pmod 4$, we must add the additional relation $(S_1RU)^3 = E$.

Here again there is the possibility that U and R alone might suffice to generate the entire group. To investigate it, we study the various subgroups of $LF(2, p^n)$ to see whether any of them can be generated by two operators satisfying the conditions $(2, 3, \{p^n-1\}/2)$. If the subgroup is commutative, or dihedral, $(p^n-1)/2$ must equal 6 or 2. If it is tetrahedral, octahedral or icosahedral, the corresponding values for $(p^n-1)/2$ are 3, 4, 5. The groups of order p^md can be eliminated by the same reasoning as was used in the Bulletin paper, after we exclude the cases $(p^n-1)/2 = 2, 3$. Linear fractional groups corresponding to Galois fields of order p^k, where k divides n are obviously impossible. Hence $(p^n-1)/2$ may not exceed 6. But n must be greater than 1. Hence the only case to consider is $p^n = 9$.

We dispose of this case at once by remembering that $LF(2, 3^2)$ can not be generated by two operators of periods two and three.* Hence, excluding this one case, U and R are sufficient to generate the entire group.

The question of the independence of the $2n+5$ conditions (or the $2n+6$ as the case may be) presents the same general difficulties that were met in the preceding considerations. Here again, however, it seemed interesting to study a special case merely to see how much improvement might be obtained in particular instances. The group chosen for study was $LF(2, 3^3)$, for which Todd's generators satisfy the relations (2, 3, 13). By the enumeration process, a complete definition was obtained by adjoining only two new conditions

$$LF(2, 3^3) \equiv (2, 3, 13), \qquad (Q^2P^5QP^9)^2 = (Q^2P^9QP^5)^2 = 1,$$

so that in this particular case six of Todd's conditions are redundant.

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* G. A. Miller, this Bulletin, vol. 7 (1900–1901), p. 426.