ON THE UNIQUENESS OF THE SOLUTIONS OF DIFFERENTIAL EQUATIONS*

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It is well known that the existence of solutions of systems of differential equations can be established under hypotheses not strong enough to guarantee uniqueness of the solution. The standard device for ensuring uniqueness is to assume that the functions involved satisfy a certain Lipschitz condition. Graves† showed that, for systems consisting of a single equation, this could be replaced by a certain monotoneity requirement. In this note we shall establish a uniqueness theorem which contains both of these as special cases.

We suppose that $f^1(x, y), \cdots, f^n(x, y)$ are functions defined for all x in an interval $[a, b]$ and all points $(y^1, \cdots, y^n)$ of n-space; each $f^i(x, y)$ is assumed continuous in y for each fixed x, and measurable in x for each fixed y. Under these conditions it can be shown‡ that if there exists a function $S(x)$ summable over $a \leqq x \leqq b$ such that§

$$| f(x, y) | \leqq S(x),$$

then, for each x_0 in $[a, b]$ and each point y_0, there is an absolutely continuous function $y(x) \equiv (y^1(x), \cdots, y^n(x))$ such that $y(x_0) = y_0$, and||

$$\dot{y}^i(x) = f^i(x, y(x)), \qquad a \leqq x \leqq b, \tag{1}$$

for almost all x. However, this solution may not be unique.¶

We therefore establish the following theorem:

THEOREM I. *Let the functions $f^i(x, y)$ be defined for all (x, y) with $a \leqq x \leqq b$. Let there exist a function $M(x)$ summable over $[a, b]$ such that for all x in $[a, b]$, all y and all η, the inequality*

$$\{f^i(x, y + \eta) - f^i(x, y)\}\eta^i \leqq M(x) | \eta |^2 \tag{2}$$

holds. Then, if $y_1(x)$ and $y_2(x)$ are absolutely continuous functions satisfying the differential equations (1) *for almost all x, and for some x_0 in*

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† L. M. Graves, *The existence of an extremum in problems of Mayer*, Transactions of this Society, vol. 39 (1936), pp. 456–471; in particular, p. 459.

‡ Carathéodory, *Vorlesungen über reelle Funktionen*, p. 672.

§ The symbol $|v|$ denotes the length of the vector v; thus $|f| = (f^i f^i)^{1/2}$.

|| The symbol $\dot{y}^i$ denotes the derivative $y^{i\prime}(x)$, where that derivative exists and is finite; elsewhere it has the value 0.

¶ Carathéodory, op. cit., p. 675.

$[a, b]$ the equation $y_1(x_0)=y_2(x_0)$ holds, these functions are identical for $x_0 \leqq x \leqq b$.

Define $\eta^i(x)=y_2^i(x)-y_1^i(x)$; then $\eta(x_0)=0$. If our theorem is false, there is a number k such that $x_0<k\leqq b$ and $|\eta(k)|>0$. Let h be the greatest value of x less than k for which $|\eta(x)|=0$; such a number surely exists, since $|\eta(x)|$ is continuous and $|\eta(x_0)|=0$. We now have

$$(3) \qquad |\eta(h)| = 0; \qquad |\eta(x)| > 0, \qquad h < x \leqq k.$$

Since the functions y_1 and y_2 satisfy the equations (1), for almost all x in $[a, b]$ the equations

$$(4) \qquad \dot{\eta}^i(x) = f^i(x, y_2) - f^i(x, y_1) = f^i(x, y_1 + \eta) - f^i(x, y_1)$$

hold; whence, by (2),

$$(5) \qquad \dot{\eta}^i\eta^i \leqq \{f^i(x, y_1 + \eta) - f^i(x, y_1)\}\eta^i \leqq M(x)|\eta|^2.$$

If we define $\lambda(x)=\log|\eta(x)|=(\log \eta^i\eta^i)/2$, this can be written (because of (3)) in the form

$$(6) \qquad \dot{\lambda}(x) \leqq M(x), \qquad h < x \leqq k.$$

On each interval $[\xi, k]$ with $h<\xi<k$, the function $|\eta(x)|$ is absolutely continuous and bounded from zero. Hence $\lambda(x)$ is also absolutely continuous on $[\xi, k]$, and from (6) we obtain by integration

$$\lambda(k) - \lambda(\xi) \leqq \int_\xi^k M(x)dx \leqq \int_a^b |M(x)|dx, \quad h < \xi \leqq k.$$

Thus $\lambda(\xi)$ is bounded below on the interval $h<\xi\leqq k$. But this is a contradiction; for, as x approaches h from the right, $|\eta(x)|$ approaches 0, and $\lambda(x)=\log|\eta(x)|$ approaches $-\infty$. Our theorem is therefore established.

COROLLARY. *Let the functions $f^i(x, y)$ be defined for all (x, y) with $a\leqq x\leqq b$. Let $y_1(x)$ and $y_2(x)$ be absolutely continuous functions satisfying equations* (1) *and coinciding at a point x_0 of the interval $[a, b]$. Let any one of the following five conditions be satisfied.*

(i) *To each point (x_0, y_0) with $a\leqq x\leqq b$ there corresponds a positive number ϵ and a function $M(x)$, summable over an interval $[\alpha, \beta]$ having $\alpha<x_0<\beta$, such that*

$$(7) \qquad \{f^i(x, y + \eta) - f^i(x, y)\}\eta^i \leqq M(x)|\eta|^2,$$

whenever $\alpha\leqq x\leqq\beta$ and $|\eta|<\epsilon$.

(ii) *The same condition as* (i) *with* (7) *replaced by*

$$(8) \qquad -\{f^i(x, y + \eta) - f^i(x, y)\}\eta^i \leqq M(x)|\eta|^2.$$

(iii) *The same as* (i) *with* (7) *replaced by*

$$|\{f^i(x, y+\eta) - f^i(x, y)\}\eta^i| \leqq M(x)|\eta|^2. \tag{9}$$

(iv)* *The same as* (i) *with* (7) *replaced by*

$$|f(x, y+\eta) - f(x, y)| \leqq M(x)|\eta|. \tag{10}$$

(v)† $n=1$, *and* $f^1(x, y)$ *is a monotonic decreasing function of* y *for each fixed* x.

Then the identity $y_1(x) \equiv y_2(x)$ *holds on the corresponding intervals*:

(i): $x_0 \leqq x \leqq b$; (iii), (iv): $a \leqq x \leqq b$;
(ii): $a \leqq x \leqq x_0$; (v): $x_0 \leqq x \leqq b$.

The proof of (i) is essentially that of Theorem 1. We need only observe that, having found the $[\alpha, \beta]$ and the ϵ belonging to the point $(h, y(h))$, we can reduce k, if necessary, so that $[h, k] \subset [\alpha, \beta]$ and $|\eta(x)| < \epsilon$ if $h \leqq x \leqq k$. Part (ii) can be established analogously. More simply, it can be obtained from (i) by the transformation $\bar{x} = -x$. Part (iii) follows by applying (i) to the interval $x_0 \leqq x \leqq b$ and (ii) to the interval $a \leqq x \leqq x_0$. If condition (iv) holds, then, by the Cauchy-Schwarz inequality,

$$\begin{aligned} |\{f^i(x, y+\eta) - f^i(x, y)\}\eta^i| &\leqq |f(x, y+\eta) - f(x, y)| \cdot |\eta| \\ &\leqq M(x)|\eta|^2; \end{aligned}$$

so (iv) is a special case of (iii). For (v), we observe that $f^1(x, y+\eta) - f^1(x, y)$ cannot have the same sign as η; so the inequality

$$\{f^1(x, y+\eta) - f^1(x, y)\}\eta \leqq M(x)|\eta|^2$$

holds with $M(x) \equiv 0$. Hence (v) is a special case of (i).

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* Carathéodory, op. cit., pp. 673–674.

† Graves, loc. cit.