NEW POINT CONFIGURATIONS AND ALGEBRAIC CURVES CONNECTED WITH THEM*

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1. **Introduction.** In the memorial volume† for Professor Hayashi, I studied an involutorial Cremona transformation in a projective S_r which is obtained as follows: Let $C_i=(ax)_i\lambda_i^2+(bx)_i\lambda_i+(cx)_i=0$, $(i=1, 2, \cdots, r)$, be r hypercones in S_r. Every value of λ_i determines a hypertangent plane to the cone C_i. Thus the parameters $\lambda_1, \lambda_2, \cdots, \lambda_r$ for the hypercones $C_1, C_2, \cdots, C_r$, in the same order, determine r hyperplanes which intersect in a point (x) of S_r. From this point (x) there pass, one for each of the r hypercones, r more tangent hyperplanes whose parameters $\lambda_1', \lambda_2', \cdots, \lambda_r'$ are in the same order uniquely determined by the set $\lambda_1, \lambda_2, \cdots, \lambda_r$, and hence are rational functions

$$\rho\lambda_i' = \phi_i(\lambda_1, \lambda_2, \cdots, \lambda_r), \qquad i = 1, 2, \cdots, r,$$

of the parameters λ. Conversely, the set $\lambda_1', \lambda_2', \cdots, \lambda_r'$ determines λ_i uniquely: $\sigma\lambda_i=\phi_i(\lambda_1', \lambda_2', \cdots, \lambda_r')$. If therefore the λ's and λ''s are interpreted as coordinates of points of euclidean spaces $E_r(\lambda)$ and $E_r'(\lambda')$, there exists an involutorial Cremona transformation between the two r-dimensional spaces. The order and fundamental elements of this involution were determined in the corresponding projective spaces S_r and S_r' and applications given for S_2 and S_3. These belong to a remarkable class of involutions which have the property that when in S_r and S_r'

$$P\ (\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_{r+1}), \qquad P'\ (\lambda_1', \lambda_2', \lambda_3', \cdots, \lambda_{r+1}')$$

are corresponding points and any number of transpositions between coordinates in the same columns is performed, say

$$Q\ (\lambda_1, \lambda_2', \lambda_3', \cdots, \lambda_i', \cdots, \lambda_r, \cdots, \lambda_{r+1}'),$$
$$Q'\ (\lambda_1', \lambda_2, \lambda_3, \cdots, \lambda_i, \cdots, \lambda_r', \cdots, \lambda_{r+1}),$$

then Q, Q' is always a couple of corresponding points of the involution.

To this class also belong the well known quadratic and cubic involutions in S_2, $\rho x_i'=1/x_i$, $(i=1, 2, 3)$, and in S_3, $\rho x_i'=1/x_i$, $(i=1, 2, 3, 4)$,

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† The Tôhoku Mathematical Journal, vol. 37 (1933), pp. 100–109. See also Commentarii Mathematici Helvetici, vol. 4 (1932), pp. 65–73.

and in general in S_r, $\rho x_i' = 1/x_i$, $(i=1, 2, \cdots, r+1)$. It is the purpose of this paper to show the importance of these in connection with the plane elliptic cubic in S_2 and a certain septimic of genus three in S_3.

2. **The Δ_8-configuration on the plane elliptic cubic.** Let A_1 (1, 0, 0); A_2 (0, 1, 0); A_3 (0, 0, 1); B (1, 1, 1); B_1 (−1, 1, 1); B_2 (1, −1, 1); B_3 (1, 1, −1) be the fundamental and invariant points of the quadratic involution in S_2, $T_2 \equiv \rho x_i' = 1/x_i$, $(i=1, 2, 3)$, and perform the possible permutations between the coordinates of corresponding points as indicated above, so that we obtain the four couples of corresponding points

$$
\begin{array}{ll}
P_1 \ (a_1, a_2, a_3), & P_2 \ (a_2a_3, a_2, a_3), \\
P_1' \ (a_2a_3, a_3a_1, a_1a_2), & P_2' \ (a_1, a_3a_1, a_1a_2), \\
P_3 \ (a_1, a_3a_1, a_3), & P_4 \ (a_1, a_2, a_1a_2), \\
P_3' \ (a_2a_3, a_2, a_1a_2), & P_4' \ (a_2a_3, a_3a_1, a_3).
\end{array}
$$

This is easily verified. Take for example P_2; its inverse is $(1/a_2a_3, 1/a_2, 1/a_3)$. Multiplying by $a_1a_2a_3$, we obtain as the same point (a_1, a_3a_1, a_1a_2) which is P_2'. The eight points of this Δ_8 configuration can be grouped into couples whose joins pass through A_1, A_2, A_3 as indicated in the following table:

$$
\begin{array}{l}
A_1\colon \ P_1P_2, \ P_1'P_2', \ P_3P_4', \ P_3'P_4; \\
A_2\colon \ P_1P_3, \ P_1'P_3', \ P_2P_4', \ P_2'P_4; \\
A_3\colon \ P_1P_4, \ P_1'P_4', \ P_2P_3', \ P_2'P_3.
\end{array}
$$

It is moreover evident that the joins of corresponding points P_iP_i' pass through the point O $(a_1+a_2a_3, a_2+a_3a_1, a_3+a_1a_2)$. The result may be stated as the following theorem:

THEOREM 1. *The eight points of a Δ_8-configuration lie by twos on four lines through each of the points A_i. The joins of the four pairs of corresponding points pass through a fixed point O, uniquely determined by one pair of corresponding points in a Δ_8.*

Now O is the isologue of an invariant elliptic cubic C_3 of the involution, uniquely determined by any pair P_iP_i' of Δ_8. Of course, C_3 passes through Δ_8, B, B_1, B_2, B_3, and also O' (a_2a_3, a_3a_1, a_1a_2). Draw any other secant s through O cutting C_3 in a pair Q_1Q_1' of corresponding points. These determine a new Δ_8-configuration on the same C_3, so that there are ∞^1 Δ_8-configurations on C_3 attached to the triangle $A_1A_2A_3$.

On any elliptic cubic C_3 there are ∞^1 Steinerian quadruples (points of tangency of four tangents from a point of C_3 to C_3) and for each a diagonal triangle $A_1A_2A_3$. Choose $A_1A_2A_3$ as the fundamental triangle* of a quadratic transformation and the Steinerian quadruple as the set of invariant points B, B_1, B_2, B_3. The elliptic cubic C_3 is invariant in this involution and the tangents at the B's cut C_3 in its isologue O. Hence we have the following theorem:

THEOREM 2. *There are ∞^1 Δ_8-configurations on each of the diagonal triangles of the ∞^1 Steinerian quadruples of a given elliptic C_3, completely inscribed in this cubic.*

3. **The Δ_{16}-configuration and a septimic of genus three in S_3 connected with it.** (1) Consider the involutorial cubic transformation T_3: $\rho x_i' = 1/x_i$, $(i=1, 2, 3, 4)$, in an S_3 with the fundamental points A_1 (1, 0, 0, 0), A_2 (0, 1, 0, 0), A_3 (0, 0, 1, 0), A_4 (0, 0, 0, 1), and the invariant points B_i $(\pm 1, \pm 1, \pm 1, \pm 1)$, $(i=1, 2, \cdots, 8)$; and perform all possible permutations, or series of transpositions as explained in §1. In this manner a configuration Δ_{16} of eight couples of corresponding points P_iP_i' is obtained as shown in the table

P_1 (a_1, a_2, a_3, a_4);	P_5 $(a_1, a_2, a_3, a_1a_2a_3)$;
P_1' $(a_2a_3a_4, a_1a_3a_4, a_1a_2a_4, a_1a_2a_3)$;	P_5' $(a_2a_3a_4, a_1a_3a_4, a_1a_2a_4, a_4)$;
P_2 $(a_2a_3a_4, a_2, a_3, a_4)$;	P_6 $(a_2a_3a_4, a_1a_3a_4, a_2, a_3)$;
P_2' $(a_1, a_1a_3a_4, a_1a_2a_4, a_1a_2a_3)$;	P_6' $(a_1, a_2, a_1a_2a_4, a_1a_2a_3)$;
P_3 $(a_1, a_1a_3a_4, a_3, a_4)$;	P_7 $(a_2a_3a_4, a_2, a_1a_2a_4, a_4)$;
P_3' $(a_2a_3a_4, a_2, a_1a_2a_4, a_1a_2a_3)$;	P_7' $(a_1, a_1a_3a_4, a_3, a_1a_2a_3)$;
P_4 $(a_1, a_2, a_1a_2a_4, a_4)$;	P_8 $(a_2a_3a_4, a_2, a_3, a_1a_2a_3)$;
P_4' $(a_2a_3a_4, a_1a_3a_4, a_3, a_1a_2a_3)$;	P_8' $(a_1, a_1a_3a_4, a_1a_2a_4, a_4)$.

As in case of S_2 it may be verified at once that the points of each pair P_iP_i' correspond, and that the sixteen points lie by twos on eight lines through each A_i. If Q is any point of S_3, then the line A_iQ is transformed into the line A_iQ'. The lines P_1P_2, P_3P_6 on A_1 and P_1P_3, P_2P_6 on A_2 form a quadrilateral in the plane $a_4x_3 - a_3x_4 = 0$; $P_1'P_2'$, $P_3'P_6'$ on A_1, and $P_1'P_3'$, $P_2'P_6'$ on A_2 a quadrilateral in the conjugate plane $a_3x_3 - a_4x_4 = 0$. The table of the thirty-two lines, eight through each A_i follows:

* For T_2, T_3 and other involutions see the author's paper *On surfaces and curves which are invariant under involutorial Cremona transformations*, American Journal of Mathematics, vol. 48 (1926), pp. 21–44.

$$A_1\colon\ P_1P_2,\ P_1'P_2',\ P_3P_6,\ P_3'P_6',\ P_4P_7,\ P_4'P_7',\ P_5P_8,\ P_5'P_8';$$

$$A_2\colon\ P_1P_3,\ P_1'P_3',\ P_2P_6,\ P_2'P_6',\ P_4P_8',\ P_4'P_8,\ P_5P_7',\ P_5'P_7;$$

$$A_3\colon\ P_1P_4,\ P_1'P_4',\ P_2P_7,\ P_2'P_7',\ P_3P_8',\ P_3'P_8,\ P_5P_6',\ P_5'P_6;$$

$$A_4\colon\ P_1P_5,\ P_1'P_5',\ P_2P_8,\ P_2'P_8',\ P_3P_7',\ P_3'P_7,\ P_4P_6',\ P_4'P_6.$$

It also appears at once that the eight joins of corresponding points P_iP_i' pass through a fixed point

$$O\ (a_1 + a_2a_3a_4,\ a_2 + a_1a_3a_4,\ a_3 + a_1a_2a_4,\ a_4 + a_1a_2a_3).$$

To sum up we have the following theorem:

THEOREM 3. *The sixteen points of Δ_{16} lie by twos on eight lines through each of the four A_i. The eight lines on each of any two of the four A_i form four quadrilaterals on the chosen two A_i, which lie in two pairs of conjugate planes with the join of the two A_i as a common axis. The joins of corresponding points of Δ_{16} pass through a fixed point O.*

(2) It is known that the system of lines joining corresponding points of T_3 form a cubic line complex Γ:

$$p_{12}p_{13}p_{23} + p_{12}p_{14}p_{42} + p_{13}p_{14}p_{34} + p_{23}p_{42}p_{34} = 0$$

so that the lines of Γ on a point $O\ (b_1, b_2, b_3, b_4)$, $b_1 = a_1 + a_2a_3a_4, \cdots$, generate the cubic complex-cone

$$\begin{aligned} K = {} & (b_1x_2 - b_2x_1)(b_1x_3 - b_3x_1)(b_2x_3 - b_3x_2) \\ & + (b_1x_2 - b_2x_1)(b_1x_4 - b_4x_1)(b_4x_2 - b_2x_4) \\ & + (b_1x_3 - b_3x_1)(b_1x_4 - b_4x_1)(b_3x_4 - b_4x_3) \\ & + (b_2x_3 - b_3x_2)(b_4x_2 - b_2x_4)(b_3x_4 - b_4x_3) = 0. \end{aligned}$$

Among the generatrices of K are the eight joins P_iP_i' of Δ_{16}. The eight lines OB_i lie on K. Any other generatrix g of K is on two corresponding points Q and Q' of T_3. These determine another Δ_{16} uniquely, which also lies on K. Thus there are ∞^1 Δ_{16}'s on K. Corresponding points QQ' on K form a certain space curve whose order is obtained as follows: The join of $Q(x)$, $Q'(x')$ passes through O when

$$\lambda x_1 + \mu x_2x_3x_4 = b_1, \qquad \lambda x_2 + \mu x_1x_3x_4 = b_2,$$

$$\lambda x_3 + \mu x_1x_2x_4 = b_3, \qquad \lambda x_4 + \mu x_1x_2x_3 = b_4.$$

Eliminating λ, μ, 1 from any two distinct triples of these equations, say between the first three and the last three, the cubic cones K_4 and K_1 with vertices A_4 and A_1 and the common generatrix A_1A_4 are obtained, along which they have the common tangent plane $b_2x_2 - b_3x_3$

$=0$. Hence they intersect in a residual septimic C_7, the locus of the point Q, Q'. This follows immediately by inspection of the equations

$$K_4 = b_1x_1(x_2^2 - x_3^2) + b_2x_2(x_3^2 - x_1^2) + b_3x_3(x_1^2 - x_2^2) = 0,$$

$$K_1 = b_2x_2(x_3^2 - x_4^2) + b_3x_3(x_4^2 - x_2^2) + b_4x_4(x_2^2 - x_3^2) = 0.$$

This can be verified by other methods of proof which for the sake of brevity shall be omitted.

(3) To prove that the genus of C_7 is three, project C_7 upon K_1 from a generic point P. The projection proper is a residual C_{14} of order $3\times 7-7=14$. The cone K_4 cuts C_{14} in $3\times 14-6=36$ points proper, because C_7 touches both K_1 and K_4 along A_1A_4 in three points which accounts for six (improper) points of intersection. The polar conic of P with respect to K_1 cuts C_7 outside of A_1 and A_4 in twelve points, so that altogether $36-12=24$ points of intersection are left which are projected into twelve double points of C_7', the projection of C_7 upon a generic plane. The genus p of C_7' and hence of C_7 is therefore $p=6\times 5/2-12=3$.

Now every couple $P_i'P_i'$ on K or C_7 gives rise to a definite Δ_{16}-configuration. Hence we have our next theorem:

THEOREM 4. *On every cubic cone K of the cubic complex associated with the involutorial cubic transformation T_3 there exists an invariant septimic C_7 of genus three with ∞^1 Δ_{16}-configurations.*

It is interesting to note that the C_7 lies on two other cubic cones K_2 and K_3 with vertices at A_2 and A_3 by using the elimination process of λ, μ, 1 in the remaining possible ways, so that it may also be characterized by the property that *it lies on five cubic cones.*

(4) The investigation may be extended to any other S_r, $r>3$, but this would amount merely to a simple generalization of the preceding theory.

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