RELATIONS AMONG THE FUNDAMENTAL SOLUTIONS OF THE GENERALIZED HYPERGEOMETRIC EQUATION WHEN $p=q+1$. I. NON-LOGARITHMIC CASES*

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It is the purpose of this paper to develop the relations among the non-logarithmic solutions of the equation

$$\left\{\prod_{t=1}^{p}(\theta+a_t)-\frac{1}{z}\theta\prod_{t=1}^{q}(\theta+c_t-1)\right\}y=0,\quad p=q+1, \tag{1}$$

where $\theta=z(d/dz)$. The results of this paper generalize those of Mehlenbacher,† who studied the case in which $p=2$ and $q=1$, and extend those of Barnes‡ who obtained the asymptotic developments of the non-logarithmic solutions of (1) for those cases in which $p<q+1$.

The succeeding analysis will be simplified if we rewrite equation (1) in the equivalent form

$$\left\{\prod_{t=1}^{q+1}(\theta+a_t)-\frac{1}{z}\prod_{t=1}^{q+1}(\theta+c_t-1)\right\}y=0,\qquad c_{q+1}=1. \tag{2}$$

If no two of the a_t or c_t are equal or differ by an integer, then equation (2) has $q+1$ linearly independent solutions about the point $z=0$, which may be written

$$Y_{0j}=z^{1-c_j}\prod_{t=1}^{q+1}\frac{\Gamma(1+c_t-c_j)}{\Gamma(1+a_t-c_j)}\sum_{n=0}^{\infty}\prod_{t=1}^{q+1}\frac{\Gamma(1+a_t-c_j+n)}{\Gamma(1+c_t-c_j+n)}z^n, \tag{3}$$
$$j=1,2,3,\cdots,q+1;\ c_{q+1}=1;\ |z|<1;$$

and $q+1$ linearly independent solutions about the point $z=\infty$ which may be written

$$Y_{\infty j}=z^{-a_j}\prod_{t=1}^{q+1}\frac{\Gamma(1-a_t+a_j)}{\Gamma(1-c_t+a_j)}\sum_{n=0}^{\infty}\prod_{t=1}^{q+1}\frac{\Gamma(1-c_t+a_j+n)}{\Gamma(1-a_t+a_j+n)}\frac{1}{z^n}, \tag{4}$$
$$j=1,2,3,\cdots,q+1;\ c_{q+1}=1;\ |z|>1.$$

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† See this Bulletin, Abstract 42-5-169.

‡ E. W. Barnes, *The asymptotic expansions of integral functions defined by generalized hypergeometric series*, Proceedings of the London Mathematical Society, (2), vol. 5 (1907), pp. 59–116.

In this paper, we shall develop the relations among solutions (3) and (4) of equation (2).

In this connection, we may state the following theorem:

THEOREM 1. *If no two of the a_t or c_t are equal or differ by an integer, then the solutions* (3) *of equation* (2), *when extended analytically outside their circle of convergence, may be expressed linearly in terms of solutions* (4) *in the following forms, it being understood throughout that* $0 < \arg z < 2\pi$:

$$Y_{0j} = \sum_{k=1}^{q+1} \left\{ e^{i\pi(a_k - c_j + 1)} \frac{\Gamma(c_j - a_k)\Gamma(1 + c_k - c_j)}{\Gamma(c_k - a_k)} \cdot \prod_{\substack{t=1 \\ t \neq k}}^{q+1} \frac{\Gamma(a_t - a_k)\Gamma(1 + c_t - c_j)}{\Gamma(c_t - a_k)\Gamma(1 + a_t - c_j)} Y_{\infty k} \right\}, \tag{5}$$

$$j = 1, 2, 3, \cdots, q + 1; c_{q+1} = 1.*$$

PROOF. It is well known that the function

$$g(w) = \prod_{t=1}^{q+1} \frac{\Gamma(1 + a_t - c_j + w)}{\Gamma(1 + c_t - c_j + w)} \tag{6}$$

satisfies the relation

$$g(w) = w^{\sum_{t=1}^{q+1}(a_t - c_t)}[1 + \gamma(w)], \qquad \lim_{|w| \to \infty} \gamma(w) = 0, \ |\arg w| < \pi.†$$

Therefore,

$$|g(w)| \leqq |w^{\sum_{t=1}^{q+1}(a_t - c_t)}| [1 + |\gamma(w)|] = |w|^R e^{-I\phi}[1 + |\gamma(w)|],$$

where R and I denote the real and imaginary parts of $\sum_{t=1}^{q+1}(a_t - c_t)$, respectively, and where $\phi = \arg w$. For sufficiently large values of $|w|$, we have $|\gamma(w)| < \epsilon$, $|\arg w| < \pi$. Thus, for such values of $|w|$

$$|g(w)| < |w|^R e^{-I\phi}(1 + \epsilon) < |w|^R e^{I\pi}(1 + \epsilon) < K|w|^R,$$

where K is a constant independent of w. The function $g(w)$, therefore, satisfies the conditions of a general theorem due to Ford,‡ but, as ap-

* As far as relation (5) itself is concerned, the restriction $c_{q+1} = 1$ is unnecessary. The restriction is necessary, however, in order that the Y_{0j} and the $Y_{\infty j}$ be solutions of equation (2).

† See, for example, Milne-Thomson, *The Calculus of Finite Differences*, Macmillan, 1933, p. 255.

‡ See Ford, *The Asymptotic Developments of Functions Defined by Maclaurin Series*, University of Michigan Press, 1936, pp. 5, 9.

pears from (6), has simple poles at the points

$$(7)\quad w = c_j - a_k - n - 1,\quad n = 0, 1, 2, \cdots;\ k = 1, 2, 3, \cdots, q + 1.$$

An application of this theorem to the function

$$(8)\qquad f(z) = \sum_{n=0}^{\infty} g(n)z^n = \sum_{n=0}^{\infty} \prod_{t=1}^{q+1} \frac{\Gamma(1 + a_t - c_j + n)}{\Gamma(1 + c_t - c_j + n)} z^n$$

gives the result

$$(9)\qquad f(z) \sim - \sum_{n=1}^{\infty} \prod_{t=1}^{q+1} \frac{\Gamma(1 + a_t - c_j - n)}{\Gamma(1 + c_t - c_j - n)} \frac{1}{z^n} - \sum_{k=1}^{q+1} \sum_{n=0}^{\infty} R_{n,k},$$
$$0 < \arg z < 2\pi,$$

where $R_{n,k}$ represents the residue of the function

$$(10)\qquad \frac{\pi(-z)^w g(w)}{\sin \pi w} = (-z)^w \Gamma(w)\Gamma(1 - w) \prod_{t=1}^{q+1} \frac{\Gamma(1 + a_t - c_j + w)}{\Gamma(1 + c_t - c_j + w)}$$

at the point $w = c_j - a_k - n - 1$. But the first series on the right in (9) drops out since the denominator of every term contains a factor for which $t = j$, and

$$\frac{1}{\Gamma(1 + c_j - c_j - n)} = \frac{1}{\Gamma(1 - n)} = 0, \qquad n = 1, 2, 3, \cdots.$$

It follows that our problem reduces to that of computing the residue term.

For this purpose, consider

$$(11)\quad R_{n,k} = \frac{1}{2\pi i} \int_{C_{n,k}} (-z)^w \Gamma(w)\Gamma(1 - w) \prod_{t=1}^{q+1} \frac{\Gamma(1 + a_t - c_j + w)}{\Gamma(1 + c_t - c_j + w)} dw,$$

where $C_{n,k}$ surrounds the pole $w = c_j - a_k - n - 1$ of (10) but no other pole of (10). If in (11) we replace w by $-(w - c_j + a_k + n + 1)$, we obtain

$$R_{n,k} = -\frac{(-z)^{c_j - a_k - n - 1}}{2\pi i}$$
$$(12)\qquad \cdot \int_{C_0} (-z)^{-w} \frac{\Gamma(c_j - a_k - w - n - 1)\Gamma(w - c_j + a_k + n + 2)\Gamma(-w - n)}{\Gamma(c_k - a_k - w - n)}$$
$$\cdot \prod_{\substack{t=1\\ t\neq k}}^{q+1} \frac{\Gamma(a_t - a_k - w - n)}{\Gamma(c_t - a_k - w - n)} dw,$$

where C_0 surrounds the origin. By several applications of the relation

$$\Gamma(1 - w) = \frac{\pi}{\Gamma(w) \sin \pi w}, \tag{13}$$

we may change (12) to the form

$$\begin{aligned} R_{n,k} = &- \frac{(-z)^{c_j-a_k-n-1}(-1)^n}{2\pi i} \\ &\cdot \int_{C_0} \frac{\pi(-z)^{-w}}{\sin \pi w} \frac{\Gamma(c_j - a_k - w)\Gamma(w - c_j + a_k + 1)}{\Gamma(c_k - a_k - w)\Gamma(w - c_k + a_k + 1)} \\ &\cdot \prod_{\substack{t=1 \\ t\neq k}}^{q+1} \frac{\Gamma(a_t - a_k - w)\Gamma(w - a_t + a_k + 1)}{\Gamma(c_t - a_k - w)\Gamma(w - c_t + a_k + 1)} \\ &\cdot \prod_{t=1}^{q+1} \frac{\Gamma(w - c_t + a_k + n + 1)}{\Gamma(w - a_t + a_k + n + 1)} dw, \end{aligned} \tag{14}$$

which by virtue of a well known theorem in the calculus of residues* is equal to

$$\begin{aligned} R_{n,k} = &- \frac{(-z)^{c_j-a_k-1}}{z^n} \frac{\Gamma(c_j - a_k)\Gamma(1 - c_j + a_k)}{\Gamma(c_k - a_k)\Gamma(1 - c_k + a_k)} \\ &\cdot \prod_{\substack{t=1 \\ t\neq k}}^{q+1} \frac{\Gamma(a_t - a_k)\Gamma(1 - a_t + a_k)}{\Gamma(c_t - a_k)\Gamma(1 - c_t + a_k)} \cdot \prod_{t=1}^{q+1} \frac{\Gamma(1 - c_t + a_k + n)}{\Gamma(1 - a_t + a_k + n)}. \end{aligned} \tag{15}$$

Therefore,

$$\begin{aligned} \sum_{n=0}^{\infty} R_{n,k} = &- e^{i\pi(a_k-c_j+1)} z^{c_j-a_k-1} \frac{\Gamma(c_j - a_k)\Gamma(1 - c_j + a_k)}{\Gamma(c_k - a_k)\Gamma(1 - c_k + a_k)} \\ &\cdot \prod_{\substack{t=1 \\ t\neq k}}^{q+1} \frac{\Gamma(a_t - a_k)\Gamma(1 - a_t + a_k)}{\Gamma(c_t - a_k)\Gamma(1 - c_t + a_k)} \cdot \sum_{n=0}^{\infty} \prod_{t=1}^{q+1} \frac{\Gamma(1 - c_t + a_k + n)1}{\Gamma(1 - a_t + a_k + n)z^n} \\ = &- e^{i\pi(a_k-c_j+1)} z^{c_j-1} \frac{\Gamma(c_j - a_k)\Gamma(1 - c_j + a_k)}{\Gamma(c_k - a_k)} \\ &\cdot \prod_{\substack{t=1 \\ t\neq k}}^{q+1} \frac{\Gamma(a_t - a_k)}{\Gamma(c_t - a_k)} Y_{\infty k}. \end{aligned} \tag{16}$$

Thus,

* See Ford, op. cit., p. 5, equation (20).

$$
(17)\qquad \sum_{k=1}^{q+1}\sum_{n=0}^{\infty} R_{n,k} = -z^{c_j-1}\sum_{k=1}^{q+1}\left\{ e^{i\pi(a_k-c_j+1)}\frac{\Gamma(c_j-a_k)\Gamma(1-c_j+a_k)}{\Gamma(c_k-a_k)} \cdot \prod_{\substack{t=1\\ t\neq k}}^{q+1}\frac{\Gamma(a_t-a_k)}{\Gamma(c_t-a_k)} Y_{\infty k}\right\}.
$$

Substituting (17) into (9) and recalling that the first series of (9) is no longer present, we have

$$
(18)\qquad f(z) \sim z^{c_j-1}\sum_{k=1}^{q+1}\left\{ e^{i\pi(a_k-c_j+1)}\frac{\Gamma(c_j-a_k)\Gamma(1-c_j+a_k)}{\Gamma(c_k-a_k)} \cdot \prod_{\substack{t=1\\ t\neq k}}^{q+1}\frac{\Gamma(a_t-a_k)}{\Gamma(c_t-a_k)} Y_{\infty k}\right\}.
$$

Since each $Y_{\infty k}$ which appears in (18) is defined by a series which converges for $|z|>1$, we may replace $\sim$ by $=$ in (18). Then, upon noting that

$$
(19)\qquad Y_{0j} = z^{1-c_j}\prod_{t=1}^{q+1}\frac{\Gamma(1+c_t-c_j)}{\Gamma(1+a_t-c_j)} f(z),
$$

we see that (18) is equivalent to the desired result (5).

By means of a similar proof, we may establish the following theorem:

THEOREM 2. *If no two of the a_t or c_t are equal or differ by an integer, then the solutions* (4) *of equation* (2), *when extended analytically outside their circle of convergence, may be expressed linearly in terms of solutions* (3) *in the following forms, it being understood throughout that* $0<\arg(1/z)<2\pi$:

$$
(20)\qquad Y_{\infty j} = \sum_{k=1}^{q+1}\left\{ e^{i\pi(a_j-c_k+1)}\frac{\Gamma(c_k-a_j)\Gamma(1-a_k+a_j)}{\Gamma(c_k-a_k)} \cdot \prod_{\substack{t=1\\ t\neq k}}^{q+1}\frac{\Gamma(c_k-c_t)\Gamma(1-a_t+a_j)}{\Gamma(c_k-a_t)\Gamma(1-c_t+a_j)} Y_{0k}\right\},
$$

$$
j = 1, 2, 3, \cdots, q+1;\ c_{q+1} = 1.
$$

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