EXTENSIONS OF FUNCTIONALS ON COMPLEX LINEAR SPACES

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The theorem of Hahn-Banach* states that any linear (continuous) functional defined on a linear subspace of a normed linear space can be extended to the whole space without increasing the norm or modulus of the functional. This theorem however deals only with real linear spaces and real-valued functionals. It is unfortunately not entirely possible to remove the restriction to real values and real coefficients. The present paper discusses this question and investigates the extent to which the theorem of Hahn-Banach remains valid in the general case of complex linear spaces. It is of interest to notice that the difficulties which appear are not due to the possibility that a linear space may be of infinite dimension; the difficulties are present essentially even in the finite dimensional case.

To simplify the exposition let us agree to call a functional $f(x)$ a *real linear* functional if, for any real numbers a, b and any two elements x, y of a linear space L, the relation

$$f(ax + by) = af(x) + bf(y)$$

holds. If the space L is complex linear, and if the above relation remains valid even for all pairs of complex numbers a, b, then the functional will be called *complex linear*. On a complex linear space we may of course have a real linear functional which is not complex linear. Let us remark that by a real subspace of a complex linear space shall be meant any subspace which contains all real linear combinations of any finite number of its elements. Such a subspace may also contain some or all complex linear combinations of its elements. A functional defined over a real subspace will be called *complex linear* if the above relation holds for all complex linear combinations which do not lead to elements outside the subspace. We are of course concerned only with the extension of complex linear functionals.

The theorem of Hahn-Banach has this analogue:

THEOREM 1. *Let l be any complex linear subspace of a normed complex linear space L. Let $f(x)$ be any complex linear functional defined on l, having a norm M. Then there always exists a complex linear functional $F(x)$ defined on L, which coincides with $f(x)$ in l, and which has the same norm M on L.*

* See Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 28 and p. 55.

PROOF. For $x\epsilon l$, let $f(x)=f_1(x)+if_2(x)$, where $f_1(x)$ and $f_2(x)$ are real-valued. It is easily verified that they are real linear functionals in l of norm $\leqq M$, and that they satisfy furthermore the relation $f_2(x)=-f_1(ix)$. Let $F_1(x)$ be a real linear extension of $f_1(x)$ according to Banach's theorem. Then the equation

$$F(x) = F_1(x) - iF_1(ix)$$

defines a complex linear functional on L, and $F(x)$ coincides with $f(x)$ in l. To verify that the norm of $F(x)$ is M, we notice that, if $F(x)=re^{i\theta}$, then

$$|F(x)| = |e^{-i\theta}F(x)| = |F_1(e^{-i\theta}x)| \leqq M\|x\|.$$

A similar proof is given in a paper by F. J. Murray,* for the case of a functional defined on a linear manifold in an L_p-space.

The subspace l in Theorem 1 is assumed to be a complex linear subspace; that this assumption is an essential one is shown by the following theorem:

THEOREM 2. *In any complex linear space L of infinite dimension, there always exists a real linear subspace l on which there is a complex linear functional $f(x)$ which cannot be continuously extended to the entire space L.*

PROOF. Let $x_1, y_2, \cdots, y_n, \cdots$ be a denumerable sequence of complex linearly independent elements of L, of norm 1, and put

$$z_n = \frac{-nx_1 + y_n}{\|-nx_1 + y_n\|}, \qquad n = 2, 3, \cdots.$$

Then the z_n's and x_1 are complex linearly independent. We construct next by complete induction a sequence $x_1, x_2, x_3, \cdots$ to satisfy the following conditions:

(1) $$x_n = \exp(i\theta_n)z_n, \qquad (\theta_n \text{ real}),$$

(2) $$\|r_1x_1 + r_2x_2 + r_3x_3 + \cdots + r_nx_n\| \geqq r_1$$

for all real numbers $r_1, \cdots, r_n$.

For $n=1$, we take simply $\theta_1=0$. Suppose now that $x_1, x_2, x_3, \cdots, x_{n-1}$ are already determined, and consider in the complex plane w the region R consisting of all complex numbers w for each of which

$$\|x_1 + r_2x_2 + \cdots + r_{n-1}x_{n-1} + wz_n\| < 1$$

* *Linear transformations in $\mathfrak{L}_p$, $p>1$*, Transactions of this Society, vol. 39 (1936), p. 84.

for a certain choice of real numbers $r_2, r_3, \cdots, r_{n-1}$. This region R, if it exists, is obviously open and convex and does not contain the origin $w=0$. Hence there will exist a straight line passing through the origin which has no point in common with R. Let $w=\exp(i\theta_n)$ be on this straight line, and put $x_n=\exp(i\theta_n)z_n$. The condition (2) is satisfied, trivially for $r_1=0$, and if $r_1\neq 0$ because of the very construction of θ_n. The subspace l, consisting of all real linear combinations of a finite number of $x_1, x_2, \cdots, x_n, \cdots$ is real linear and satisfies the further property that the element

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

does not belong to l if any one c_i is not real. Thus the functional

$$f(x) = c_1$$

is a (real-valued) complex linear functional in l, and is of norm 1 by the property (2) which determined the sequence $\{x_n\}$. This functional cannot be extended to the space L, for if $F(x)$ denotes a complex linear extension and M its norm, we may consider the element

$$nx_1 + \|-nx_1 + y_n\| \exp(-i\theta_n)x_n = y_n.$$

The norm of this element equals 1, and yet the value of $F(x)$ for it must be n, which implies a contradiction if $n>M$.

For a given subspace l of finite dimension, the extension is always possible, although the norm may have to be multiplied by a constant >1. The construction of the above proof shows that no finite upper bound for this constant exists if the space L is of dimension $\geqq 2$.

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