CONCERNING SETS OF POLYNOMIALS ORTHOGONAL SIMULTANEOUSLY ON SEVERAL CIRCLES*

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1. **Introduction.** Starting with the well-known fact that the set of polynomials $\{z^n\}$ is an orthogonal set on every circle $|z|=R$, there has recently been some consideration of the general problem suggested, that of the existence of sets of polynomials in the complex variable which are orthogonal, with respect to suitable norm functions, simultaneously on more than one curve. Terming a set of polynomials $\{p_n(z)\}$ *canonical* on a rectifiable Jordan curve C with respect to the positive continuous norm function $n(z)$ provided it is found by orthogonalizing on C the set $\{z^n\}$ with respect to $n(z)$, and provided the coefficient of z^n in $p_n(z)$ is chosen positive, we list the previous results† which we shall find pertinent to our purpose:

(1) Walsh‡ and Szegö§ have shown, independently and by different methods, that if the same set of polynomials is canonical on two distinct curves then one of the curves is a "level curve" (Kreisbild)‖ in the conformal mapping of the region outside the other curve onto the exterior of a circle, the points at infinity corresponding to each other.

(2) Szegö¶ has exhibited *all* canonical sets of polynomials in the complex variable, each set canonical on *all* level curves of a given family; there are only five essentially different types of such sets.

While this last result is definitive in connection with sets of polynomials canonical simultaneously on a whole family of level curves, the general problem of the existence of sets of polynomials canonical simultaneously on only a *finite* number of curves has not yet been discussed; in the references cited above, Walsh (p. 136) and Szegö (p. 196) both suggest its study. It is the purpose of the present paper

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† A complete list of results on this problem will be found in §1 of Walsh and Merriman, *Note on the simultaneous orthogonality of harmonic polynomials on several curves*, Duke Mathematical Journal, vol. 3 (1937), pp. 279–288.

‡ J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, American Mathematical Society Colloquium Publications, vol. 20, p. 134, Theorem 11.

§ G. Szegö, *A problem concerning orthogonal polynomials*, Transactions of this Society, vol. 37 (1935), pp. 196–206, Theorem I.

‖ Cf. Walsh, loc. cit., §4.1.

¶ Loc. cit., pp. 197–198.

to consider this problem in the special case that the curves involved are concentric (see (1) above) circles with center at the origin. To state our results briefly, it is shown that if a set of polynomials in the complex variable is a canonical set *simultaneously on two distinct concentric circles* $|z|=R_j>r$, $(j=1,\ 2)$, *then the set is canonical on all circles* $|z|=R>r$ *concentric with these. Thus, there exist no sets of polynomials in the complex variable canonical simultaneously on only a finite number* (>1) *of concentric circles.*

2. **Preliminaries.** We consider the set of polynomials

$$(1) \qquad p_0(z) = 1,\ p_n(z) = a_0^{(n)} + a_1^{(n)}z + \cdots + a_{n-1}^{(n)}z^{n-1} + z^n, \qquad n = 1, 2, \cdots .$$

The orthogonality criterion used is

$$(2) \qquad \int_{|z|=R} p_k(z)\overline{p_l(z)}n(z)\,|\,dz\,| = 0, \qquad k \neq l,$$

where* $n(z)$ denotes the real, positive, continuous norm function, alterable by multiplication by a positive constant. The function $n(z)$ can be expressed as $n(z)=|D(z)|^2$, where $D(z)$ is analytic and nonvanishing outside a basic circle $|z|=r$, the point $z=\infty$ included. Since $D(z)$ depends only on the polynomials $p_n(z)$, orthogonality on several circles yields one and the same $D(z)$; thus, with allowance for the possible multiplicative constant, we must have $n_j(z)=|D(z)|^2$, $(j=1, 2, \cdots)$, on each circle of orthogonality $|z|=R_j$, $(j=1, 2, \cdots)$. The whole configuration being studied may be subjected to a linear integral transformation.

The method used is that of determining, from the hypothesis of orthogonality simultaneously on the two distinct circles $|z|=R_j$, $(j=1,\ 2)$, necessary values of the coefficients $a_i^{(n)}$, $(i=0, 1, \cdots, n-1;\ n=1, 2, \cdots)$, of the polynomials (1) and a necessary form for a suitable norm function on the two circles. This determination leads to some one of the sets of polynomials which are known to be canonical† on *all* circles $|z|=R>r$:

* The following facts concerning $n(z)$ are digested from Szegö, loc. cit., pp. 197, 201.

† Szegö, loc. cit. We have for our own subsequent purposes recorded slightly more general sets II than those given by Szegö, resulting from a permissible linear integral transformation $z=a^{-1/\alpha}z'$ of his sets; an irrelevant constant has been removed from each $p_n(z)$. Sets II, $a=1$, were exhibited by Szegö (Mathematische Annalen, vol. 79 (1919), pp. 323–339) without mention of orthogonality on more than one curve, and by Walsh (Mémorial des Sciences Mathématiques, no. 73 (1935), p. 43) with mention of orthogonality on all circles $|z|=R>|a|^{1/\alpha}$.

I. *On the circles* $|z|=R>0$, $D(z)=1$, $p_n(z)=z^n$;

II. *On the circles* $|z|=R>|a|^{1/\alpha}$, α *a positive integer*, $a\neq 0$ *a complex constant*, $D(z)=(1-az^{-\alpha})^{-1}$, $p_n(z)=z^n$ *for* $0\leqq n<\alpha$, $p_n(z)=z^{n-\alpha}(z^\alpha-a)$ *for* $n\geqq\alpha$.

The set I is the limiting set of the sets II as $\alpha\to\infty$; the orthogonality of I, even on the circles $|z|=R$, $0<R\leqq|a|^{1/\alpha}$, occurs since in such a limiting process $D(z)$ becomes constant, hence analytic in the whole plane. We thus confine our attention to the case $D(z)\not\equiv 1$, that is, $D(z)$ analytic outside $|z|=r$, the point at infinity included. We shall choose r as the smaller, say R_1, of R_1 and R_2.

3. **Two auxiliary formulas.** We shall use the following form for $n(z)$:

$$n(z) = 1 + \sum_{j=1}^{\infty} A_j z^j + \sum_{j=1}^{\infty} \overline{A}_j \bar{z}^j. \tag{3}$$

This development (3) is simply the formal Fourier development of the real function $n(z)$ rearranged by means of the equations, valid on $|z|=R$,

$$z^n = R^n(\cos n\theta + i\sin n\theta), \qquad \bar{z}^n = R^n(\cos n\theta - i\sin n\theta),$$

$$\sin n\theta = (z^n - \bar{z}^n)/2iR^n, \qquad \cos n\theta = (z^n + \bar{z}^n)/2R^n.$$

Of course, to obtain the form (3) we have multiplied the preceding development by the reciprocal of its first term, which is positive.

This form of the norm function of sets II is, for z on $|z|=R>|a|^{1/\alpha}$,

$$n(z) = 1 + \sum_{j=1}^{\infty} R^{-2j\alpha}\bar{a}^j z^{j\alpha} + \sum_{j=1}^{\infty} R^{-2j\alpha}a^j\bar{z}^{j\alpha}. \tag{3'}$$

Use of (3) permits calculation of a set of important auxiliary integrals:

$$\int_{|z|=R} z^k\bar{z}^l n(z)\,|dz| = \begin{cases} 2\pi\overline{A}_{k-l}R^{2k+1}, & k>l,\\ 2\pi R^{2k+1}, & k=l,\\ 2\pi A_{l-k}R^{2l+1}, & k<l.\end{cases} \tag{4}$$

For example, if $k>l$,

$$\int_{|z|=R} z^k\bar{z}^l n(z)\,|dz| = R^{2l}\int_{|z|=R} z^{k-l}n(z)\,|dz|$$

$$= R^{2l}\overline{A}_{k-l}\int_{|z|=R} z^{k-l}\bar{z}^{k-l}\,|dz| = R^{2k}\overline{A}_{k-l}\int_{|z|=R}|dz| = 2\pi R^{2k+1}\overline{A}_{k-l},$$

by use of the familiar $\int_{|z|=R} z^p \bar{z}^q |dz| = 0$, $p \neq q$. The term-by-term integration is justifiable since the Fourier development of $n(z)$, continuous on $|z|=R$, is uniformly summable (Cesàro) to $n(z)$ on $|z|=R$.

4. **The first steps in an induction.** We proceed to the first steps in an inductive proof. First we have, using (4),

$$\int_{|z|=R} p_1(z)\overline{p_0(z)}n(z)\,|dz| = \int_{|z|=R} (a_0^{(1)} + z)n(z)\,|dz|$$
$$= 2\pi R[a_0^{(1)} + \bar{A}_1 R^2] = 0$$

for* $R=R_j$, $(j=1, 2)$. Hence $a_0^{(1)} = -\bar{A}_1 R^2$. Since $a_0^{(1)}$ is the same constant for the two values of R, we must have $\bar{A}_1 R^2 = k_1$, with k_1 a complex constant; thus $A_1 = \bar{k}_1 R^{-2}$, necessarily, on the two circles. We separate these findings into two cases:

$$(5)\qquad \begin{cases} \text{(a)} & k_1 \neq 0;\ a_0^{(1)} = -k_1,\ p_1(z) = z - k_1,\ A_1 = \bar{k}_1 R^{-2}; \\ \text{(b)} & k_1 = 0;\ a_0^{(1)} = 0,\quad p_1(z) = z,\quad A_1 = 0. \end{cases}$$

We continue with (5a):

$$\int_{|z|=R} p_2(z)\overline{p_0(z)}n(z)\,|dz| = \int_{|z|=R} [a_0^{(2)} + a_1^{(2)}z + z^2]n(z)\,|dz|$$
$$= 2\pi R[a_0^{(2)} + a_1^{(2)}\bar{A}_1 R^2 + \bar{A}_2 R^4] = 0,$$

so that we have, using (5a),

$$(6)\qquad a_0^{(2)} + k_1 a_1^{(2)} + \bar{A}_2 R^4 = 0.$$

Similarly, with use of (6),

$$\int_{|z|=R} p_2(z)\overline{p_1(z)}n(z)\,|dz| = \int_{|z|=R} \bar{z}[a_0^{(2)} + a_1^{(2)}z + z^2]n(z)\,|dz|$$
$$= 2\pi R[a_0^{(2)} A_1 R^2 + a_1^{(2)} R^2 + \bar{A}_1 R^4] = 0,$$

which yields the pair of equations

$$a_0^{(2)} k_1 R_j^{-2} + a_1^{(2)} + k_1 = 0, \qquad j = 1, 2.$$

These have the unique solution $a_0^{(2)} = 0$, $a_1^{(2)} = -k_1$, which, with (6), gives $A_2 = \bar{k}_1^2 R_j^{-4}$, $(j=1, 2)$. We add, then, to (5a) the following information:

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* Henceforth, results involving R will be considered tacitly to be evaluated and valid only for $R=R_j$, $(j=1, 2)$.

(7) $k_1 \neq 0;\ p_1(z) = z - k_1,\ p_2(z) = z^2 - k_1 z,\ A_1 = \bar{k}_1 R^{-2},\ A_2 = \bar{k}_1^2 R^{-4}.$

On the basis of (7) we proceed to find cumulatively

$$\int_{|z|=R} p_3(z)\overline{p_0(z)}n(z)\,|dz| = 2\pi R\big[a_0^{(3)} + a_1^{(3)}\bar{A}_1 R^2 + a_2^{(3)}\bar{A}_2 R^4 + \bar{A}_3 R^6\big] = 0, \tag{8}$$

$$\int_{|z|=R} p_3(z)\overline{p_1(z)}n(z)\,|dz| = 2\pi R\big[a_0^{(3)} A_1 R^2 + a_1^{(3)} R^2 + a_2^{(3)}\bar{A}_1 R^4 + \bar{A}_2 R^6\big] = 0, \tag{9}$$

$$\int_{|z|=R} p_3(z)\overline{p_2(z)}n(z)\,|dz| = 2\pi R\big[a_0^{(3)} A_2 R^4 + a_1^{(3)} A_1 R^4 + a_2^{(3)} R^4 + \bar{A}_1 R^6\big] = 0. \tag{10}$$

From (10) and (9) we obtain the four equations

$$\begin{cases} a_0^{(3)}\bar{k}_1^2 + a_1^{(3)}\bar{k}_1 R_j^2 + a_2^{(3)} R_j^4 + k_1 R_j^4 = 0, & j = 1, 2, \\ a_0^{(3)}\bar{k}_1 + a_1^{(3)} R_j^2 + a_2^{(3)} k_1 R_j^2 + k_1^2 R_j^2 = 0, & j = 1, 2. \end{cases} \tag{11}$$

Of these we select the first three, the augmented matrix of which system is

$$M\colon \left\|\begin{matrix} \bar{k}_1^2 & \bar{k}_1 R_1^2 & R_1^4 & -k_1 R_1^4 \\ \bar{k}_1^2 & \bar{k}_1 R_2^2 & R_2^4 & -k_1 R_2^4 \\ \bar{k}_1 & R_1^2 & k_1 R_1^2 & -k_1^2 R_1^2 \end{matrix}\right\|.$$

We first show that the determinant d_{123} (the subscripts indicate columns of M making up the determinant) is not zero; the method used is chosen to illustrate a later general discussion. We multiply the last row of d_{123} by $\bar{k}_1 \neq 0$ and subtract the first row from the result; then a Laplace development according to the two-rowed determinants of the first two rows yields, aside from an irrelevant non-zero constant multiplier, $d_{123} = (R_2^2 - R_1^2)(|k_1|^2 R_1^2 - R_1^4)$. Thus, since $R_1 \neq R_2$, $d_{123} = 0$ when and only when $|k_1| = R_1$; but in this event we can interchange the rôles of R_1 and R_2 in the discussion to obtain a new determinant which, treated in the same manner as the original, can never vanish because now $|k_1| \neq R_2$.

The determinants d_{134} and d_{234} are zero because of the proportionality of the last two columns of M. The determinant $d_{124} = -k_1 d_{123}$. Hence the unique solution of the first three equations of (11) is $a_2^{(3)} = -k_1$, $a_0^{(3)} = a_1^{(3)} = 0$; these satisfy the fourth equation of (11). Used in connection with (8) these results produce $A_3 = \bar{k}_1^3 R_j^{-6}$,

($j=1$, 2). We incorporate these findings with (7) in the necessary forms

$$(12)\qquad \begin{cases} k_1 \neq 0;\ p_1(z) = z - k_1,\ p_2(z) = z^2 - k_1 z,\ p_3(z) = z^3 - k_1 z^2, \\ A_1 = \bar{k}_1 R^{-2},\ A_2 = \bar{k}_1^2 R^{-4},\ A_3 = \bar{k}_1^3 R^{-6}. \end{cases}$$

These are the first three polynomials and A's in set* II, $\alpha=1$, $a=k_1$. Here, of course, for complete identification, in fact for meaning, we must choose $|k_1|^{1/\alpha}=|k_1|<R_1$ (cf. (3')).

5. **Continuation.** We return to (5b). We now have

$$\int_{|z|=R} p_2(z)\overline{p_0(z)}n(z)\,|\,dz\,| = a_0^{(2)} + \overline{A}_2 R^4 = 0,$$

$$\int_{|z|=R} p_2(z)\overline{p_1(z)}n(z)\,|\,dz\,| = a_1^{(2)} R^2 = 0.$$

Hence $a_1^{(2)}=0$, $a_0^{(2)}=-\overline{A}_2R^4=-k_2$, k_2 a complex constant. The following two possibilities are then to be added to (5b):

$$(13)\begin{cases} \text{(a)}\ \ k_1 = 0,\ k_2 \neq 0;\ p_1(z) = z,\ p_2(z) = z^2 - k_2,\ A_1 = 0,\ A_2 = \bar{k}_2 R^{-4}; \\ \text{(b)}\ \ k_1 = k_2 = 0;\quad p_1(z) = z,\ p_2(z) = z^2,\qquad A_1 = A_2 = 0. \end{cases}$$

Pursuing first (13a) we have the following replacements for (8), (9), and (10):

$$(14)\qquad a_0^{(3)} + a_2^{(3)} k_2 + \overline{A}_3 R_j^6 = 0, \qquad j = 1, 2,$$

$$(15)\qquad a_1^{(3)} R_j^2 + k_2 R_j^2 = 0, \qquad j = 1, 2,$$

$$(16)\qquad a_0^{(3)} \bar{k}_2 + a_2^{(3)} R_j^4 = 0, \qquad j = 1, 2.$$

From the last four equations we choose the system formed by (16) and the first equation of (15); its augmented matrix is

$$\begin{Vmatrix} \bar{k}_2 & 0 & R_1^4 & 0 \\ \bar{k}_2 & 0 & R_2^4 & 0 \\ 0 & R_1^2 & 0 & -k_2R_1^2 \end{Vmatrix}.$$

Again $d_{123}\neq 0$, but d_{124} and d_{234} are zero, while $d_{134}=k_2d_{123}$. Thus $a_0^{(3)}=a_2^{(3)}=0$, $a_1^{(3)}=-k_2$, uniquely, and (14) gives $A_3=0$. We continue (13a) as

* That is, the polynomials and A's with subscripts 1, 2, 3; here and later we omit $p_0(z)$ and A_0 from the list since they are unity in all cases.

$$\begin{aligned}(17)\qquad & k_1 = 0,\ k_2 \neq 0;\ p_1(z) = z,\ p_2(z) = z^2 - k_2,\ p_3(z) = z^3 - k_2 z,\\ & A_1 = A_3 = 0,\ A_2 = \bar{k}_2 R^{-4}.\end{aligned}$$

These are the first three polynomials and A's of set II, $\alpha=2$, $a=k_2$, $|k_2|^{1/\alpha}=|k_2|^{1/2}<R_1$.

6. **Continuation.** Returning to (13b), we now have the equations $a_0^{(3)}+\bar{A}_3R_j^6=0$, $a_1^{(3)}R_j^2=0$, $a_2^{(3)}R_j^4=0$, $(j=1, 2)$, which have the immediate unique solution $a_1^{(3)}=a_2^{(3)}=0$, $a_0^{(3)}=-\bar{A}_3R^6=-k_3$, so that (13b) is continued as

$$(18)\quad \begin{cases} \text{(a)}\quad k_1 = k_2 = 0,\ k_3 \neq 0;\ p_1 = z,\ p_2 = z^2,\ p_3 = z^3 - k_3; \\ \qquad\qquad\qquad\qquad\qquad\qquad A_1 = A_2 = 0,\ A_3 = k_3 R^{-6}; \\ \text{(b)}\quad k_1 = k_2 = k_3 = 0;\quad p_1 = z,\ p_2 = z^2,\ p_3 = z^3; \\ \qquad\qquad\qquad\qquad\qquad\qquad A_1 = A_2 = A_3 = 0. \end{cases}$$

In (18a) *we have the first three polynomials and A's in set* II, $\alpha=3$, $a=k_3$, $|k_3|^{1/\alpha}=|k_3|^{1/3}<R_1$. *In* (18b) *we have the first three polynomials and A's in all* sets* II, $\alpha>3$.

7. **The general step.** For the general step in the inductive proof we follow the lead given by (12), (17), and (18) and the italicized statements following them, and assume that, on the hypothesis of orthogonality on the two distinct circles $|z|=R_j$, $(j=1, 2)$, the first m polynomials of (1) have been found, on these two circles, to be necessarily identical with the first m polynomials in some specific one, say (σ), of the sets II, $\alpha\leqq m$, or with the first m polynomials in all sets II, $\alpha>m$, while the norm function has been determined to the same extent of coincidence. We should then expect to show that the hypothesis implies that the first $m+1$ polynomials of (1) must be, on the two circles, identical with the first $m+1$ polynomials in the specified set (σ) of II, $\alpha\leqq m$, or in the set II, $\alpha=m+1$, or in all the sets II, $\alpha>m+1$,† while the A_{m+1} obtained in the process is the one anticipated. These expectations are realized; we outline the proof.

We first evaluate the integrals

$$\int_{|z|=R} p_{m+1}(z)\overline{p_i(z)}n(z)\,|dz| = 0, \qquad i = 0, 1, 2, \cdots, m.$$

* The first m polynomials and A's in *all* sets II, $\alpha>m$, are the same.

† The last two possibilities compose the "split" case analogous to (18) which, we recall, may lead in the limit, $\alpha\to\infty$, to case I.

Here $p_{m+1}(z)$ is the $(m+1)$th polynomial in (1), $p_i(z)$, $(i=1, 2, \cdots, m)$, are the first m polynomials after $p_0(z)=1$ in some arbitrary set (σ) of II, $\alpha \leqq m$, or in all sets II, $\alpha > m$, and

$$
\begin{aligned}
n(z) = {} & 1 + 0 + \cdots + 0 + R^{-2\alpha}\bar{a}z^{\alpha} + \cdots + R^{-2q\alpha}\bar{a}^q z^{q\alpha} \\
& + 0 + \cdots + 0 + R^{-2\alpha}a\bar{z}^{\alpha} + \cdots + R^{-2q\alpha}a^q\bar{z}^{q\alpha} \\
& + \sum_{j=m+1}^{\infty} A_j z^j + \sum_{j=m+1}^{\infty} \overline{A}_j \bar{z}^j;
\end{aligned}
\tag{19}
$$

in (19) $q\alpha$ is the largest integral multiple of α not greater than m, and all A's with subscripts between multiples of α are zero; also we assume, for meaning, $|a|^{1/\alpha} < R_1$. The cumulative results of the integrations are the $2(m+1)$ equations (we have retained the vanishing A's for symmetry of writing):

$$
\left\{
\begin{aligned}
& a_0^{(m+1)} + a_1^{(m+1)}\overline{A}_1R_j^2 + a_2^{(m+1)}\overline{A}_2R_j^4 + \cdots + \overline{A}_{m+1}R_j^{2(m+1)} = 0, \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, \\
& a_0^{(m+1)}A_1R_j^2 + a_1^{(m+1)}R_j^2 + a_2^{(m+1)}\overline{A}_1R_j^4 + \cdots + \overline{A}_mR_j^{2(m+1)} = 0, \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, \\
& a_0^{(m+1)}A_2R_j^4 + a_1^{(m+1)}A_1R_j^4 + a_2^{(m+1)}R_j^4 + \cdots + \overline{A}_{m-1}R_j^{2(m+1)} = 0, \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& a_0^{(m+1)}A_mR_j^{2m} + a_1^{(m+1)}A_{m-1}R_j^{2m} + \cdots + a_m^{(m+1)}R_j^{2m} \\
& \qquad\qquad\qquad + \overline{A}_1R_j^{2(m+1)} = 0, \qquad j = 1, 2.
\end{aligned}
\right.
\tag{20}
$$

Omitting the first pair of equations in (20) since it contains the unknown $\overline{A}_{m+1}$ as well as the unknowns $a_i^{(m+1)}$, $(i=0, 1, \cdots, m)$, and reading the other pairs in reverse order, we select the pair whose leading term is $a_0^{(m+1)}A_{q\alpha}R_j^{2q\alpha}$, $(j=1, 2)$, and the first one of each other pair. Thus we have a system of $m+1$ linear non-homogeneous equations in the $m+1$ unknowns $a_i^{(m+1)}$, $(i=0, 1, \cdots, m)$. The augmented matrix, M, of this system is displayed in skeleton form on page 65. In the left-hand part of M we have indicated the diagonal panels of zeros arising from the vanishing A's in $n(z)$; in the right-hand part the zeros have not been inserted, the interest here being on the subscripts of the A's in the final columns.

Our general argument is valid unless $\alpha > m$, when the first and last columns of M consist exclusively of zeros; this special case we post-

$$
\begin{array}{r}
\\
m-q\alpha\ \text{rows}\left\{\begin{array}{l} \\ \\ \\ \\ \end{array}\right. \\
\\
\\
\\
\alpha-1\ \text{rows}\left\{\begin{array}{l} \\ \\ \\ \\ \\ \end{array}\right. \\
\\
\\
\\
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\\
\alpha-1\ \text{rows}\left\{\begin{array}{l} \\ \\ \\ \\ \end{array}\right.
\end{array}
\begin{array}{cccccccccccccccccccc}
\multicolumn{5}{c}{\overbrace{\hspace{6em}}^{m-q\alpha\ \text{columns}}} & \multicolumn{5}{c}{\overbrace{\hspace{6em}}^{\alpha-1\ \text{columns}}} \\
0 & \cdot & \cdot & 0 & A_{q\alpha}R_1^{2m} & 0 & \cdot & \cdot & \cdot & 0 & A_{(q-1)\alpha}R_1^{2m} & \cdot & \cdot & A_{\alpha-1}R_1^{2m} & \cdot & \cdot & \cdot & R_1^{2m} & -\overline{A}_1R_1^{2(m+1)} \\
\cdot & \cdot & 0 & A_{q\alpha}R_1^{2(m-1)} & 0 & \cdot & \cdot & \cdot & 0 & A_{(q-1)\alpha}R_1^{2(m-1)} & 0 & \cdot & \cdot & A_{\alpha-2}R_1^{2(m-1)} & \cdot & \cdot & R_1^{2(m-1)} & \overline{A}_1R_1^{2m} & -\overline{A}_2R_1^{2(m+1)} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \multicolumn{5}{c}{\overbrace{\hspace{6em}}^{\alpha-1\ \text{columns}}} \\
A_{q\alpha}R_1^{2q\alpha} & 0 & \cdot & \cdot & \cdot & 0 & A_{(q-1)\alpha}R_1^{2q\alpha} & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & R_1^{2q\alpha} & \cdot & \overline{A}_{m-q\alpha}R_1^{2m} & -\overline{A}_{m-q\alpha+1}R_1^{2(m+1)} \\
A_{q\alpha}R_2^{2q\alpha} & 0 & \cdot & \cdot & \cdot & 0 & A_{(q-1)\alpha}R_2^{2q\alpha} & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & R_2^{2q\alpha} & \cdot & \overline{A}_{m-q\alpha}R_2^{2m} & -\overline{A}_{m-q\alpha+1}R_2^{2(m+1)} \\
0 & \cdot & \cdot & \cdot & 0 & A_{(q-1)\alpha}R_1^{2(q\alpha-1)} & 0 & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\overline{A}_{m-q\alpha+2}R_1^{2(m+1)} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & A_1R_1^{2(m-\alpha)} & \cdot & \cdot & \cdot & \cdot & -\overline{A}_{\alpha-1}R_1^{2(m+1)} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & R_1^{2(m-\alpha+1)} & \cdot & \multicolumn{2}{c}{[(\alpha+1)\text{th row}]} & \cdot & -\overline{A}_{\alpha}R_1^{2(m+1)} \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \overline{A}_1R_1^{2(m-\alpha+1)} & \cdot & \cdot & \cdot & \cdot & -\overline{A}_{\alpha+1}R_1^{2(m+1)} \\
A_{(q-1)\alpha}R_1^{2(q-1)\alpha} & \cdot & \cdot & \cdot & \cdot & \cdot & A_{(q-2)\alpha}R_1^{2(q-1)\alpha} & \cdot & \cdot & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
A_{\alpha}R_1^{2\alpha} & 0 & \cdot & \cdot & \cdot & 0 & R_1^{2\alpha} & \cdot & \cdot & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
A_2R_1^4 & A_1R_1^4 & R_1^4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot & \overline{A}_{m-\alpha-1}R_1^{2(m-\alpha+1)} & \cdot & \cdot & \cdot & \overline{A}_{m-2}R_1^{2m} & -\overline{A}_{m-1}R_1^{2(m+1)} \\
A_1R_1^2 & R_1^2 & \overline{A}_1R_1^4 & \cdot & \cdot & \cdot & 0 & \overline{A}_{\alpha}R_1^{2(\alpha+1)} & \cdot & & & \cdot & \overline{A}_{m-\alpha}R_1^{2(m-\alpha+1)} & \cdot & \cdot & \cdot & \overline{A}_{m-1}R_1^{2m} & -\overline{A}_mR_1^{2(m+1)} \\
 & & & & & & \uparrow & & & & & & \uparrow \\
 & & & & & & (\alpha+1)\text{th column} & & & & & & (m+2-\alpha)\text{th column}
\end{array}
$$

THE MATRIX M

pone for the present. If $\alpha \leqq m$ it can be shown that d, the determinant formed by the first $m+1$ columns of M, is not zero; the proof of this fact we also postpone briefly.

Assuming, then, that $d \neq 0$, we replace the A's by their values as given in (19). The last and the $(m-\alpha+2)$th columns of M are then seen* to be in the proportion $-a:1$; thus we obtain the following unique solution for the system chosen from (20):

$$\overset{(m+1)}{a_{m+1-\alpha}} = -a, \quad \overset{(m+1)}{a_0} = \overset{(m+1)}{a_1} = \cdots = \overset{(m+1)}{a_{m-\alpha}} = \overset{(m+1)}{a_{m-\alpha+2}} = \cdots = \overset{(m+1)}{a_m} = 0.$$

When we substitute these values in the first equation of (20), it becomes

$$-a\overline{A}_{m+1-\alpha}R^{2(m+1-\alpha)} + \overline{A}_{m+1}R^{2(m+1)} = 0. \tag{21}$$

Hence we find $\overline{A}_{m+1}=0$ if $m+1$ is not a multiple of α, but if $m+1=q'\alpha$, q' a positive integer, we obtain from (21) $a \cdot a^{q'-1} = \overline{A}_{m+1}R^{2q'\alpha}$, or $\overline{A}_{m+1} = \overline{A}_{q'\alpha} = R^{-2q'\alpha}a^{q'}$. These results satisfy all the equations of (20). Moreover, they are the expected results; and the general step in the induction proof is complete except for the two postponements.

8. **Non-zeroness of d.** We exhibit d in skeleton form on page 67; the elements there indicated by an asterisk we shall call "stop-elements"; they may be considered to involve $\bar{a}^0$ or a^0, stopping the preceding descending sequence of powers of $\bar{a}$ in a given row or starting the subsequent ascending sequence of powers of a.

The proof that $d \neq 0$ involves in general three steps.

(i) We first multiply the (reading from the bottom) αth, 2αth, $\cdots$, $[(q-1)\alpha]$th rows by the non-zero numbers $\bar{a}^{q-1}$, $\bar{a}^{q-2}$, $\cdots$, $\bar{a}$, respectively, so that each row now starts with 0 or $\bar{a}^q$. Leaving intact the $(m-q\alpha+1)$th and $(m-q\alpha+2)$th rows, those symmetric in R_1 and R_2, we subtract the first of these from each remaining row which starts with the element $\bar{a}^q$ (this step is unnecessary if $\alpha > m/2$). Then the first and the $(\alpha+1)$th columns have zeros everywhere except in the two rows left intact; and in each of the new rows all elements are zero except the replacements of former non-zero elements after the stop-elements, which now are, aside from an irrelevant non-zero constant multiplier, of the form

$$|a|^r R_1^s - R_1^t, \tag{22}$$

* In the last row, for example, $\bar{A}_m$ and $\bar{A}_{m-\alpha}$ are zero together unless $m=q\alpha$; then $\bar{A}_m R_1^{2(m+1)} = a^q R_1^2$ and $\bar{A}_{m-\alpha} R_1^{2(m-\alpha+1)} = a^{q-1} R_1^2$.

$m-q\alpha$ columns $\qquad$ $\alpha-1$ columns

$$
\begin{array}{l|cccccccccccccccccc}
 & 0 & \cdot & \cdot & 0 & \bar a^{q}R_1^{2(m-q\alpha)} & 0 & \cdot & \cdot & \cdot & 0 & \bar a^{q-1}R_1^{2(m-q\alpha+\alpha)} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & {}^{*}R_1^{2m} \\
m-q\alpha & 0 & \cdot & 0 & \bar a^{q}R_1^{2(m-q\alpha-1)} & 0 & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & {}^{*}R_1^{2(m-1)} & 0 \\
\text{rows} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & 0 & \bar a^{q}R_1^{2} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & \multicolumn{5}{c}{\alpha-1 \text{ columns}} & & \multicolumn{5}{c}{\alpha-1 \text{ columns}} & & & & & \\
 & \bar a^{q} & 0 & \cdot & \cdot & \cdot & 0 & \bar a^{q-1}R_1^{2\alpha} & 0 & \cdot & \cdot & \cdot & 0 & \bar a^{q-2}R_1^{4\alpha} & \cdot & {}^{*}R_1^{2q\alpha} & \cdot & \cdot & \cdot \\
 & \bar a^{q} & 0 & \cdot & \cdot & \cdot & 0 & \bar a^{q-1}R_2^{2\alpha} & 0 & \cdot & \cdot & \cdot & 0 & \bar a^{q-2}R_2^{4\alpha} & \cdot & {}^{*}R_2^{2q\alpha} & \cdot & \cdot & \cdot \\
 & 0 & \cdot & \cdot & \cdot & 0 & \bar a^{q-1}R_1^{2(\alpha-1)} & 0 & \cdot & \cdot & \cdot & 0 & \bar a^{q-2}R_1^{4\alpha-2} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\alpha-1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & [(\alpha+2)\text{th row}] & \cdot & 0 & aR_1^{2(m-\alpha)} \\
\text{rows} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & aR_1^{2(m-\alpha-1)} & 0 \\
 & 0 & \bar a^{q-1}R_1^{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \bar a^{q-1} & 0 & \cdot & \cdot & \cdot & 0 & \bar a^{q-2}R_1^{2\alpha} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \bar a & 0 & \cdot & \cdot & \cdot & 0 & {}^{*}R_1^{2\alpha} & 0 & \cdot & \cdot & \cdot & 0 & aR_1^{2\alpha} & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & 0 & \cdot & \cdot & \cdot & 0 & {}^{*}R_1^{2(\alpha-1)} & 0 & \cdot & \cdot & \cdot & 0 & aR_1^{2(\alpha-1)} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\alpha-1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\text{rows} & 0 & 0 & {}^{*}R_1^{4} & 0 & \cdot & \cdot & \cdot & 0 & aR_1^{4} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & 0 & {}^{*}R_1^{2} & 0 & \cdot & \cdot & \cdot & 0 & aR_1^{2} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & \multicolumn{5}{c}{\alpha-1 \text{ columns}} & & & & & & & & & & \\
 & & & & & & & \uparrow & & & & & & & & & & & \\
 & & & & & & & (\alpha+1)\text{th column} & & & & & & & & & & &
\end{array}
$$

THE DETERMINANT d

(* indicates "stop-element")

where $t=\alpha r+s$. We develop d in the Laplace manner according to the two-rowed determinants of the two intact rows; again omitting an irrelevant non-zero constant factor, we thus reveal d as the product of U, the two-rowed determinant formed by the first and $(\alpha+1)$th columns of the intact rows, and its $(m-1)$-square algebraic complement d', since the algebraic complement of any other two-rowed determinant of the intact rows has one column of zeros. But $U=\bar{a}^{2q-1}(R_2^{2\alpha}-R_1^{2\alpha})$ cannot vanish because $R_1\neq R_2$, so that $d=0$ *when and only when* $d'=0$.

(ii) If $m-q\alpha\neq 0$, there is in the upper left-hand corner of d' an $(m-q\alpha)$-square array, U', of which the secondary diagonal, reading from lower left to upper right, consists of successive even powers of R_1, beginning with R_1^2, each multiplied by $\bar{a}^q$; all other elements of U' are zero, thus $U'\neq 0$. Since $m-q\alpha\leqq\alpha-1$, we can produce repetitions of U' in each block of $\alpha-1$ rows and columns reading down the left-hand side of d' by multiplying by suitable powers of $\bar{a}$ the rows repeating the pattern of powers of R_1 in U'. Leaving intact the rows containing U', we subtract each of them from each of the later rows now identical with it in its first $m-q\alpha$ elements. As a result, every element below U' in the first $m-q\alpha$ columns is zero; beyond the $(m-q\alpha)$th column zeros occur consistently in the altered rows until the stop-element is passed, while former non-zero elements beyond that element are replaced by elements of the form (22) and all others are still zero. A Laplace development of this transformed d' by the $(m-q\alpha)$-rowed determinants of the first $m-q\alpha$ columns reveals that d' is essentially the product of $U'\neq 0$ and its algebraic complement d'', of $q\alpha-1$ rows and columns; thus $d'=0$ *when and only when* $d''=0$.

(iii) In the upper left-hand corner of d'' occurs a square array, U'', of $\alpha-1-(m-q\alpha)$ rows and columns, which is like U' in structure. We now follow a procedure similar to that in (ii), replacing U' by U'' in that argument. A final Laplace expansion according to the $[\alpha-1-(m-q\alpha)]$-rowed determinants of the first $\alpha-1-(m-q\alpha)$ columns reveals that d'', essentially the product of $U''\neq 0$ by its algebraic complement d''', of $m-\alpha$ rows and columns, *vanishes when and only when* d''' *vanishes*. But d''' has a secondary diagonal consisting entirely of elements of form (22) with $r=2$, while all elements above that diagonal are zero; thus $d'''=0$ *when and only when* $|a|^{1/\alpha}=R_1$. This last is impossible under our original choice* of

* As a matter of fact, we can also circumvent the case $|a|^{1/\alpha}=R_1$ by the argument employed in connection with the determinant d_{123} in §4.

$|a|^{1/\alpha}<R_1$; hence $d'''\neq 0$. Thus $d\neq 0$ as was to be proved.*

9. **The case $\alpha>m$.** We have only to discuss the case $\alpha>m$. Here $d=0$ since the first column is all zeros, so the preceding argument fails. But in this case the A_j, $(j=1, 2, \cdots, m)$, are all zero and the equations (20) after the first pair yield $a_j^{(m+1)}=0$, $(j=1, 2, \cdots, m)$. The first pair, now

$$a_0^{(m+1)} + \overline{A}_{m+1}R_j^{2(m+1)} = 0, \qquad j = 1, 2,$$

produces in the usual way $\overline{A}_{m+1}=k_{m+1}R_j^{-2(m+1)}$, k_{m+1} a complex constant, $a_0^{(m+1)}=-k_{m+1}$. Since k_{m+1} may, or may not, be zero, we have the "split" case previously mentioned; in the former event we have the first $m+1$ polynomials and A's in II, $\alpha=m+1$, in the latter event we have the first $m+1$ polynomials and A's in *all* sets II, $\alpha>m+1$.

10. **Conclusion.** The general step in the induction is now established and the proof of the form of (1) and (3) is complete. We have proved that if (1) is a canonical set simultaneously on *two* distinct circles $|z|=R_j$, $(j=1, 2; R_1<R_2)$, then *on those circles* the set (1) must be identical with some set, (σ), of II, $|a|^{1/\alpha}<R_1$, or with I. But then, since (σ) and I are canonical sets on *all* circles $|z|=R>|a|^{1/\alpha}$, we have established the fact that (1) *is a canonical set simultaneously on all circles* $|z|=R>|a|^{1/\alpha}$ *if it is a canonical set on two such circles.*

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* The first step in this proof is unnecessary if $\alpha>m/2$, the second if $m-q\alpha=0$, the third if $m-q\alpha=\alpha-1$. The cases $m=14$, $\alpha=4, \cdots, 8$ are apt illustrations of all the various possibilities.