NEO-SYLVESTER CONTRACTIONS AND THE SOLUTION OF SYSTEMS OF LINEAR EQUATIONS*

BY F. L. WREN

1. *Introduction.* In 1851 Sylvester† stated, without proof, a theorem on contraction of determinants which had previously been stated by Hermite (1849)‡ for a special case. A proof of Sylvester's theorem was given by Studnička in 1879.§ A restatement of this theorem in a slightly different form permits significant applications to the determination of the ranks of matrices, the solution of systems of linear equations, and the calculation of partial and multiple coefficients of correlation.

2. *Restatement of Sylvester's Theorem.* The restatement of the theorem is as follows:

THEOREM 1. *In the nth order determinant,*

$$D \equiv |a_{ij}|, \qquad (i, j = 1, 2, \cdots, n),$$

let

$$M_{kl} \equiv \begin{vmatrix} a_{k_1l_1} & a_{k_1l_2} \cdots a_{k_1l_r} \\ a_{k_2l_1} & a_{k_2l_2} \cdots a_{k_2l_r} \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ a_{k_rl_1} & a_{k_rl_2} \cdots a_{k_rl_r} \end{vmatrix}$$

be a non-vanishing minor of order r, and associate with each element a_{ij} $(i \neq k, j \neq l)$ *an* $(r+1)$*-rowed minor of D defined by the identity*

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† J. J. Sylvester, *On the relation between the minor determinants of linearly equivalent quadratic functions*, Philosophical Magazine, (4), vol. 1 (1851), pp. 295–305, 415.

‡ C. Hermite, *Sur une question relative à la théorie des nombres*, Journal de Mathématiques, vol. 14 (1849), pp. 21–30.

§ F. J. Studnička, *Ueber eine neue Determinantentransformation*, Sitzungsberichte Geschichte der Wissenschaften, vol. 9 (1879), pp. 487–494.

$$D_{ij} \equiv \begin{vmatrix} a_{ij} & a_{il_1} & a_{il_2} & \cdots & a_{il_r} \\ a_{k_1j} & a_{k_1l_1} & a_{k_1l_2} & \cdots & a_{k_1l_r} \\ a_{k_2j} & a_{k_2l_1} & a_{k_2l_2} & \cdots & a_{k_2l_r} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{k_rj} & a_{k_rl_1} & a_{k_rl_2} & \cdots & a_{k_rl_r} \end{vmatrix}$$

in which M_{kl} is the first minor corresponding to a_{ij}. We have, then, the relation

$$(-1)^\kappa \; M_{kl}^{n-r-1} D = \Delta,$$

$$\kappa = k_1 + k_2 + \cdots + k_r + l_1 + l_2 + \cdots + l_r,$$

where Δ is a determinant of order $(n-r)$ defined by the identity

$$\Delta \equiv \begin{vmatrix} D_{i_1j_1} & D_{i_1j_2} & \cdots & D_{i_1j_{n-r}} \\ D_{i_2j_1} & D_{i_2j_2} & \cdots & D_{i_2j_{n-r}} \\ \cdot & \cdot & \cdot & \cdot \\ D_{i_{n-r}j_1} & D_{i_{n-r}j_2} & \cdots & D_{i_{n-r}j_{n-r}} \end{vmatrix}, \quad (i \neq k, j \neq l),$$

*in which the $(i_1, i_2, \cdots, i_{n-r})$ represent the rows and $(j_1, j_2, \cdots, j_{n-r})$ the columns of the minor of D complementary to M_{kl}.**

COROLLARY. *Attach to the determinant D a row whose elements are the sums of the elements in each respective column of D, and a column whose elements are the sums of the elements in each respective row of D. If the elements of the added row and column be treated in the same manner as the elements a_{ij}, $(i\neq k, j\neq l)$, were treated in the theorem, then the values obtained will be the sums of the elements D_{ij} in the corresponding column or row of Δ.*

The determinant D with the attached row and column is

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* For a proof of this theorem see G. Kowalewski, *Einführung in die Determinantentheorie*, pp. 90–93. Kowalewski points out that the theorem holds true even for $M_{kl}=0$. Attention is also called to the fact that when the minor M_{kl} is a single element the expansion is the same as that given by Chió (1853) in his *Mémoire sur les fonctions connues sous le nom de résultantes ou de déterminants.*

$$
(1) \qquad
\begin{array}{c|cccc|c|}
 & & & & & S_r \\
 & a_{11} & a_{12} & \cdots & a_{1n} & \sum_{j=1}^{n} a_{1j} \\
 & a_{21} & a_{22} & \cdots & a_{2n} & \sum_{j=1}^{n} a_{2j} \\
 & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & a_{n1} & a_{n2} & \cdots & a_{nn} & \sum_{j=1}^{n} a_{nj} \\
\hline
S_c & \sum_{i=1}^{n} a_{i1} & \sum_{i=1}^{n} a_{i2} & & \sum_{i=1}^{n} a_{in} & \\
\hline
\end{array} .
$$

If the elements in the S_r-column and the S_c-row are treated as the elements a_{ij}, then the resulting Δ, with its corresponding S_r-column and S_c-row, is

$$
\begin{array}{c|cccc|c|}
 & & & & & S_r \\
 & D_{11} & D_{12} & \cdots & D_{1n} & D_{r_1} \\
 & D_{21} & D_{22} & \cdots & D_{2n} & D_{r_2} \\
 & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & D_{n1} & D_{n2} & \cdots & D_{nn} & D_{r_n} \\
\hline
S_c & D_{c_1} & D_{c_2} & \cdots & D_{c_n} & \\
\hline
\end{array} ,
$$

where the D_{ij} are defined for $i \neq k$ and $j \neq l$, and

$$
D_{r_i} \equiv \begin{vmatrix}
\sum_{j=1}^{n} a_{ij} & a_{il_1} & a_{il_2} & \cdots & a_{il_r} \\
\sum_{j=1}^{n} a_{k_1 j} & a_{k_1 l_1} & a_{k_1 l_2} & \cdots & a_{k_1 l_r} \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\sum_{j=1}^{n} a_{k_r j} & a_{k_r l_1} & a_{k_r l_2} & \cdots & a_{k_r l_r}
\end{vmatrix} .
$$

By subtracting from the first column of D_{r_i} the sum of the remaining columns, we have

$$D_{r_i} = \sum_j D_{ij},$$

where the summation on j is extended over the numbers $1, 2, \cdots, n$, excluding $l_1, l_2, \cdots, l_r$.

A similar argument gives

$$D_{c_j} = \sum_i D_{ij},$$

where the summation on i is over the numbers $1, 2, \cdots, n$, excluding $k_1, k_2, \cdots, k_r$. (It might be added that, for all practical purposes, either the S_c row or the S_r column is sufficient in itself for a satisfactory relative check.)

3. *The Rank of a Matrix.* If a matrix is not of zero rank there is at least one element $a_{kl} \neq 0$.

If, in the $m \times n$ matrix

$$M \equiv \begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{Vmatrix}$$

of rank $\neq 0$, an element $a_{kl} \neq 0$, called the *pivotal element*, be chosen and a new matrix be formed whose elements are the second order determinants

$$D_{ij} = \begin{vmatrix} a_{ij} & a_{il} \\ a_{kj} & a_{kl} \end{vmatrix}, \qquad (i \neq k, j \neq l),$$

then a new matrix

$$M' \equiv \begin{Vmatrix} D_{11} & D_{12} & \cdots & D_{1n} \\ D_{21} & D_{22} & \cdots & D_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ D_{m1} & D_{m2} & \cdots & D_{mn} \end{Vmatrix}$$

will be formed. This matrix will have $(m-1)$ rows and $(n-1)$ columns, since there are no elements in the kth row and the lth column. Furthermore, the elements D_{ij} are all of the possible non-vanishing 2-rowed minors of M of which a_{kl} is a first minor.

Hence if all the $D_{ij}=0$, the matrix M is of rank 1. If there is an element $D_{rs}\neq 0$, proceed as in the previous case, using D_{rs} as the pivotal element to construct the $(m-2)\times(n-2)$ matrix

$$M'' = \begin{Vmatrix} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1n} \\ \Delta_{21} & \Delta_{22} & \cdots & \Delta_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \Delta_{m1} & \Delta_{m2} & \cdots & \Delta_{mn} \end{Vmatrix},$$

where the elements are as follows:

$$\Delta_{pq} = \begin{vmatrix} D_{pq} & D_{ps} \\ D_{rq} & D_{rs} \end{vmatrix} = \begin{Vmatrix} \begin{vmatrix} a_{pq} & a_{pl} \\ a_{kq} & a_{kl} \end{vmatrix} & \begin{vmatrix} a_{ps} & a_{pl} \\ a_{ks} & a_{kl} \end{vmatrix} \\ \begin{vmatrix} a_{rq} & a_{rl} \\ a_{kq} & a_{kl} \end{vmatrix} & \begin{vmatrix} a_{rs} & a_{rl} \\ a_{ks} & a_{kl} \end{vmatrix} \end{Vmatrix}, \quad \begin{pmatrix} p \neq k, r \\ q \neq l, s \end{pmatrix}.$$

Hence, by Theorem 1,

$$\Delta_{pq} = a_{kl}\cdot \begin{vmatrix} a_{pq} & a_{ps} & a_{pl} \\ a_{rq} & a_{rs} & a_{rl} \\ a_{kq} & a_{ks} & a_{kl} \end{vmatrix}, \quad \begin{pmatrix} p \neq k, r \\ q \neq l, s \end{pmatrix}.$$

Thus it is seen that the determinants Δ_{pq} are, except for the non-vanishing factor a_{kl}, the possible non-vanishing third-order minors of the original matrix M of which the non-vanishing second-order minor D_{rs} is a first minor. Hence, if all $\Delta_{pq}=0$, the rank of M is 2.

If there is a $\Delta_{pq}\neq 0$ the process may be repeated, this time using the elements in M''. This application, by use of Theorem 1, will give a matrix whose elements will be

$a_{kl}\cdot D_{rs}\times$(all possible non-vanishing third order minors of M of which D_{rs} is a first minor).

Repeated applications of the above contraction process, which will be called a *neo-Sylvester contraction*, give the following result:

THEOREM 2. *A matrix is of rank r if r neo-Sylvester contractions reduce it to a zero matrix.*

COROLLARY. *If, in an* $m\times n$ *matrix* $(m\leqq n)$, *the result of m neo-Sylvester contractions is a non-zero matrix, the rank of the original matrix is m.*

The check established in the Corollary to Theorem 1 also applies in the treatment of matrices.

4. *Systems of Simultaneous Linear Equations.* The augmented matrix of a system of m linear equations in n unknowns is the $m\times(n+1)$ matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & k_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & k_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} & k_m \end{pmatrix}.$$

Since the coefficients a_{ij} appear simultaneously in the determinants of both the coefficient and augmented matrices, it is obvious that the ranks of these two matrices can be obtained simultaneously by selecting for each contraction a pivotal element from the coefficient matrix.

THEOREM 3. *A system of m linear equations in n unknowns is inconsistent if and only if a series of neo-Sylvester contractions, with pivotal elements chosen each time from the coefficient matrix, will reduce the augmented matrix to a matrix with zeros everywhere except in the column of known terms* k_i.

Each such contraction of the matrix of the system of linear equations is equivalent to eliminating the unknown that corresponds to the column of coefficients from which each successive pivotal element is chosen. Consequently, when the system of equations is a consistent system of rank r, the result of $(r-1)$ neo-Sylvester contractions is the non-zero matrix of a system of $(m-r+1)$ equations in $(n-r+1)$ unknowns in which the corresponding coefficients are proportional. Any one of these equations may be solved for one of the $(n-r+1)$ unknowns in terms of the remaining $(n-r)$. This value may be substituted in one of the equations obtained by $(r-2)$ contractions. Continued substitution in the next previous system of equations results in the final determination of r of the unknowns in terms of the remaining $(n-r)$ unknowns. These values will satisfy the remaining $(m-r)$ equations of the resultant $(m-r+1)$ linearly dependent system obtained by the $(r-1)$ neo-Sylvester contractions.

THEOREM 4. *If* $(r-1)$ *neo-Sylvester contractions reduce the augmented matrix of a system of* m *equations in* n *unknowns to a non-zero matrix of dimensions* $(m-r+1)\times(n-r+2)$ *in which the elements of any one row are proportional to the corresponding elements of every other row, then* r *of the unknowns can be determined uniquely in terms of the remaining* $n-r$ *unknowns. These values will also satisfy the remaining* $m-r$ *equations.*

If $r=m=n$, we obtain the following result:

THEOREM 5. *If* $(n-1)$ *neo-Sylvester contractions reduce the augmented matrix of a system of* n *equations in* n *unknowns to a* 1×2 *non-zero matrix, then the equations have a unique solution.**

Each contraction is equivalent to the elimination of variable x_j if the pivotal element is chosen in the jth column. Hence, if the pivotal elements for the $(n-1)$ contractions be chosen from columns other than the column of coefficients of x_r and the column of constant terms, the value for x_r is obtained by dividing the final element in the column of constant terms by the final element in the column of coefficients of x_r. This is obvious, since the final element in the x_r column is, except for a sign and multiplicative factor, the expansion of the determinant of coefficients, and the element in the column of known terms k_i is, except for the same sign and multiplicative factor, the expansion of the determinant obtained from the determinant of the coefficients by replacing the elements of the rth column by the column of k's.

COROLLARY. *In the* $n\times(n+1)$ *matrix*

$$M \equiv \begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1n} & a_{1n+1} \\ a_{21} & a_{22} & \cdots & a_{2n} & a_{2n+1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} & a_{nn+1} \end{Vmatrix}$$

let $(n-1)$ *neo-Sylvester contractions be made by choosing the pivotal element for each contraction from any column other than the* j*th or* k*th columns. The quotient of the final element in the* k*th*

* A numerical example of the truth of this theorem and the use of the check suggested in connection with Theorem 1 is found in H. G. Deming's article, *A systematic method for the solution of simultaneous linear equations*, American Mathematical Monthly, vol. 35 (1928), pp. 360–363.

column by the final element in the jth column is, except for sign, the quotient of the two determinants of order n,

$$\begin{vmatrix} a_{11} & a_{12}\cdots & a_{1k}\cdots & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2k}\cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2}\cdots & a_{nk}\cdots & a_{nn} \end{vmatrix} \div \begin{vmatrix} a_{11} & a_{12}\cdots & a_{1j}\cdots & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2j}\cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2}\cdots & a_{nj}\cdots & a_{nn} \end{vmatrix},$$

in which the elements are all alike except those of the jth and kth columns. The sign factor is $(-1)^{|k-j|-1}$ *where* $|k-j|-1$ *is the number of inversions necessary to place the kth column in the dividend in the same relative position as the jth column in the divisor.*

The truth of this corollary is readily seen. The contractions would produce in the jth column of M the expansion of the determinant which is the divisor in the Corollary except for a multiplicative factor, which would be the product of the $(n-1)$ pivotal elements chosen for the successive contractions, and for a possible sign factor. The same contractions would produce in the kth column of M, except for the same sign and multiplicative factor, the expansion of the determinant which is the dividend in the Corollary multiplied by the additional sign factor $(-1)^{|k-j|-1}$, which is the number of inversions necessary to place the kth column of the dividend in the same relative position to the remaining columns as the jth column of the divisor bears to these same columns.

5. *Partial Correlation.* In the n-variable correlation problem the regression equations are

$$\sum_j r_{ij}\beta_{pj\cdot 12\cdots(p)\cdots(j)\cdots n} = r_{ip}, \tag{2}$$

$(i, j = 1, 2, \cdots, n$ except that neither has the value $p)$,

where r_{ij} are ordinary product-moment coefficients of correlation between variable i and j (hence $r_{ij}=r_{ji}$ and $r_{ii}=1$) and $\beta_{pj\ 12\cdots(p)\cdots(j)\cdots n}$ are the regression coefficients that minimize the error in estimating variable p from a linear function of the remaining $(n-1)$ variables.* These are $(n-1)$ linear equations in $(n-1)$ unknowns and hence the use of neo-Sylvester con-

* Truman L. Kelly, *Partial and multiple correlation*, in H. L. Rietz, *Handbook of Mathematical Statistics*, 1924, pp. 139–141.

tractions affords a systematic technique for determining the values of the β's; furthermore the check used in solving linear equations may be used here.

The process may be further systematized by making use of the nth order determinant of the product-moment coefficients

$$\Delta = \begin{vmatrix} 1 & r_{12} & r_{13} \cdots r_{1n} \\ r_{12} & 1 & r_{23} \cdots r_{2n} \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ r_{1n} & r_{2n} & r_{3n} \cdots 1 \end{vmatrix},$$

where Δ_{pq} will represent the minor of order $(n-1)$ obtained from Δ by deleting the pth row and qth column.

It is to be noted that the determinant of the coefficients of the β's in the regression equations (2) is the determinant Δ_{pp} and that the known terms are elements from the pth column of Δ. Perform $(n-2)$ neo-Sylvester contractions on Δ in which the pivotal element is chosen each time as any element in the principal diagonal except in the pth or qth row and column. The result of these $(n-2)$ contractions will be

$$(3) \qquad \begin{array}{r} \\ p\text{-row} \\ q\text{-row} \end{array} \begin{vmatrix} p\text{-column} & q\text{-column} \\ K\Delta_{qq} & (-1)^{|p-q|-1}K\Delta_{qp} \\ (-1)^{|p-q|-1}K\Delta_{pq} & K\Delta_{pp} \end{vmatrix},$$

where K is a common sign and multiplicative factor. Since Δ is symmetric and the pivotal element for each contraction is chosen from the principal diagonal of Δ, it follows that $\Delta_{pq} = \Delta_{qp}$; furthermore the sign factor may be written $(p+q-1)$, since, for all integral values of p and q, $(p+q-1)$ is even or odd according as $|p-q|-1$ is even or odd. Hence the determinant (3) may be written in the simpler form

$$(4) \qquad \begin{array}{r} \\ p\text{-row} \\ q\text{-row} \end{array} \begin{vmatrix} p\text{-column} & q\text{-column} \\ K\Delta_{qq} & (-1)^{p+q-1}K\Delta_{pq} \\ (-1)^{p+q-1}K\Delta_{pq} & K\Delta_{pp} \end{vmatrix}.$$

Therefore, by the Corollary to Theorem 5,

$$(5) \qquad \begin{aligned} \beta_{pq\cdot 12\cdots n} &= \frac{(-1)^{p+q-1}K\Delta_{pq}}{K\Delta_{pp}} = \frac{(-1)^{p+q-1}\Delta_{pq}}{\Delta_{pp}}, \\ \beta_{qp\cdot 12\cdots n} &= \frac{(-1)^{p+q-1}K\Delta_{qp}}{K\Delta_{qq}} = \frac{(-1)^{p+q-1}\Delta_{pq}}{\Delta_{qq}}. \end{aligned}$$

The values of the remaining $\beta_{pj\cdot 12\cdots(p)\cdots(j)\cdots n}$ can be determined as in §4. Furthermore, if the first pivotal element is chosen in the kth row and column the following β's may be obtained as described:

$\beta_{ij\cdot k}$: Divide the element in the ith row and jth column by the element in the jth row and column.

$\beta_{ji\cdot k}$: Use the process symmetrical to the one above, that is, divide the element in the jth row and ith column by the element in the ith row and column.

If the second pivotal element is chosen from the lth row and column, $\beta_{ij\cdot kl}$ and $\beta_{ji\cdot kl}$ may be obtained as above. Each subsequent contraction will produce the conjugate regression coefficients of i on j and j on i with the additional variables controlled corresponding to the row and column in which each subsequent pivotal element is chosen.

The partial coefficient of correlation $r_{pq\cdot 12\cdots(p)\cdots(q)\cdots n}$ has the value $(-1)^{p+q-1}\Delta_{pq}/(\Delta_{pp}\Delta_{qq})^{1/2}$.* Hence the value of $r_{pq\cdot 12\cdots(p)\cdots(q)\cdots n}$ may be obtained directly from determinant (4). Furthermore, $r_{ij\cdot k}$ may be obtained after the first contraction using the element in the kth row and column as the pivotal element. It is obtained, as above, from the two rowed minor of the contracted determinant obtained by deleting all rows and columns except the ith and jth. Similarly, after each subsequent contraction, values may be obtained for

$$r_{ij\cdot kl},\ r_{ij\cdot klm},\ r_{ij\cdot klmn},\ \cdots .$$

Furthermore, each contraction can be checked by attaching to Δ a column whose elements are the sums of the elements of each respective row of Δ and applying the same contraction to these elements that is applied to the elements of Δ.

6. *Multiple Correlation.* The multiple alienation coefficient is given by

$$k^2_{p\cdot 12\cdots(p)\cdots n} = \frac{\Delta}{\Delta_{pp}}.\dagger \tag{6}$$

If we define the determinant $\Delta\ (p)$ by

* Truman L. Kelley, *Statistical Methods*, p. 298.

† Ibid., p. 301.

$$\Delta(p) \equiv \begin{vmatrix} 1 & r_{12} & r_{13} \cdots 0 \cdots r_{1n} \\ r_{12} & 1 & r_{23} \cdots 0 \cdots r_{2n} \\ \cdot & \cdot & \cdot \cdots \cdot \cdots \cdot \\ r_{1p} & r_{2p} & r_{3p} \cdots 1 \cdots r_{pn} \\ \cdot & \cdot & \cdot \cdots \cdot \cdots \cdot \\ r_{1n} & r_{2n} & r_{3n} \cdots 0 \cdots 1 \end{vmatrix},$$

where the elements of the pth column of Δ have been replaced by zeros except for the element 1 in the pth row and column, it is evident that

$$\Delta(p) = \Delta_{pp}.$$

Formula (6) may now be written

$$k^2_{p\cdot 12\cdots(p)\cdots n} = \frac{\Delta}{\Delta(p)}. \tag{7}$$

If the determinant of the numerator be augmented by adding the pth column of the denominator, the $n\times(n+1)$ matrix

$$\begin{Bmatrix} 1 & r_{12} & r_{13} \cdots r_{1p} \cdots r_{1n} & 0 \\ r_{12} & 1 & r_{23} \cdots r_{2p} \cdots r_{2n} & 0 \\ \cdot & \cdot & \cdot \cdots \cdot \cdots \cdot & \cdot \\ r_{1p} & r_{2p} & r_{3p} \cdots 1 \cdots r_{pn} & 1 \\ \cdot & \cdot & \cdot \cdots \cdot \cdots \cdot & \cdot \\ r_{1n} & r_{2n} & r_{3n} \cdots r_{pn} \cdots 1 & 0 \end{Bmatrix} \tag{8}$$

is obtained. If this added column be called the p'-column of the matrix, and if the pivotal elements for the neo-Sylvester contractions be chosen each time from columns other than the p- and p'-columns, then by the Corollary to Theorem 5, after $(n-1)$ such contractions the value of $k^2_{p\cdot 12\cdots(p)\cdots n}$ can be obtained by dividing the element in the p-column by the element in the p'-column. Furthermore, the result of each contraction can be checked by the method of the Corollary to Theorem 1.

The multiple correlation coefficient is given by the relation

$$r^2_{p\cdot 12\cdots(p)\cdots n} = 1 - k^2_{p\cdot 12\cdots(p)\cdots n}. \tag{9}$$

Substituting in (9) from (7) and simplifying, we have

$$r^2_{p\cdot 12\cdots(p)\cdots n} = \frac{\Delta(p) - \Delta}{\Delta(p)}$$

$$(10)\qquad = \frac{\begin{vmatrix} 1 & r_{12} & r_{13}\cdots & -r_{1p}\cdots & r_{1n} \\ r_{12} & 1 & r_{23}\cdots & -r_{2p}\cdots & r_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ r_{1p} & r_{2p} & r_{3p}\cdots & 0\cdots & r_{pn} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ r_{1n} & r_{2n} & r_{3n}\cdots & -r_{pn}\cdots & 1 \end{vmatrix}}{\Delta(p)}.$$

Thus, by augmenting the numerator of (10) as in the case of (7), the pth column of the denominator becomes the p'th column of the matrix

$$(11)\qquad \begin{pmatrix} 1 & r_{12} & r_{13}\cdots & -r_{1p}\cdots & r_{1n} & 0 \\ r_{12} & 1 & r_{23}\cdots & -r_{2p}\cdots & r_{2n} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ r_{1p} & r_{2p} & r_{3p}\cdots & 0\cdots & r_{pn} & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ r_{1n} & r_{2n} & r_{3n}\cdots & -r_{pn}\cdots & 1 & 0 \end{pmatrix}.$$

Hence the same process and check which was applied to matrix (8) to find $k^2_{p\cdot 12\cdots(p)\cdots n}$, when applied to matrix (11), will give $r^2_{p\cdot 12\cdots(p)\cdots n}$. Furthermore, if the rows and columns containing the pivotal elements are chosen in the sequence $1, 2, 3, \cdots, n$ (omitting p) and if, after each contraction, the element in the pth row and column be divided by the element in the pth row and p'th column, then the values $k^2_{p\cdot 1}, k^2_{p\cdot 12}, k^2_{p\cdot 123}, \cdots$, and $r^2_{p\cdot 1}, r^2_{p\cdot 12}, r^2_{p\cdot 123}, \cdots$ will be obtained from matrices (8) and (11) respectively.

George Peabody College for Teachers