A NOTE ON YOUNG-STIELTJES INTEGRALS*

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THEOREM 1. *If $f(x)$ is bounded and measurable Borel, and $g_1(x)$, $g_2(x)$ are of bounded variation, then the following equality holds:*

$$\int_0^1 f(x)d[g_1(x)g_2(x)] = \int_0^1 f(x)g_1(x+0)dg_2(x) + \int_0^1 f(x)g_2(x-0)dg_1(x). \tag{1}$$

PROOF. In a recent article Evans† showed that if $g_1(x)$ and $g_2(x)$ have no common points of discontinuity, then

$$\int_0^1 f(x)d[g_1(x)g_2(x)] = \int_0^1 f(x)g_1(x)dg_2(x) + \int_0^1 f(x)g_2(x)dg_1(x).$$

Therefore (1) holds if either $g_1(x)$ or $g_2(x)$ are continuous. It remains to show that the theorem holds when $g_1(x)$ and $g_2(x)$ are both step functions. Under these circumstances we have

$$\begin{aligned}\int_0^1 & f(x)g_1(x+0)dg_2(x) + \int_0^1 f(x)g_2(x-0)dg_1(x)\\ &= \sum f(\alpha_i)g_1(\alpha_i+0)[g_2(\alpha_i+0) - g_2(\alpha_i-0)]\\ &\quad + \sum f(\alpha_i)g_2(\alpha_i-0)[g_1(\alpha_i+0) - g_1(\alpha_i-0)]\\ &= \sum f(\alpha_i)[g_1(\alpha_i+0)g_2(\alpha_i+0) - g_1(\alpha_i-0)g_2(\alpha_i-0)]\\ &= \int_0^1 f(x)d[g_1(x)g_2(x)],\end{aligned}$$

where the summations are taken over all the discontinuities of $g_1(x)$ and $g_2(x)$.

The following lemmas are immediate applications of equation (1).

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† G. C. Evans, *Correction and addition to "Complements of potential theory,"* American Journal of Mathematics, vol. 57 (1935), pp. 623–626.

LEMMA 1. *If $f(x)$ is positive, bounded, and Borel measurable, and $g_1(x)$, $g_2(x)$ are monotone increasing, bounded, continuous on the left, then*

$$\int_0^1 f(x)d[g_1(x)g_2(x)] \leqq \int_0^1 f(x)g_1(x)dg_2(x) + \int_0^1 f(x)g_2(x)dg_1(x). \tag{2}$$

LEMMA 2. *If in Lemma 1 the $g_1(x)$ and $g_2(x)$ are monotone decreasing functions continuous on the right, the inequality sign in (2) is reversed.*

W. C. Randels* used Lemma 1 in proving the existence of a solution of

$$f(x) = m(x) + \lambda \int_0^x f(y)dK(x, y).$$

In essentially the same manner, by making use of Lemma 2, we may prove the following theorem.

THEOREM 2. *If $g(x)$ is of bounded variation and if* (a) $K(x, y)$ *is Borel measurable in y for every x,* (b) $K(x+0, y) = K(x, y)$, (c) $K(x, x) = 0$, (d) $|K(x_1, y) - K(x_2, y)| \leqq |T(x_1) - T(x_2)|$, *where the function $T(x)$ is bounded and non-decreasing with x, then*

$$f(x) = g(x) + \lambda \int_0^x K(x, y)df(y) \tag{3}$$

has a unique solution of bounded variation.

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* W. C. Randels, *On Volterra-Stieltjes integral equations*, Duke Mathematical Journal, vol. 1 (1935), pp. 538–542.