any point of $U_m(b)$. Then $ax<2^{-m}$ and $xy<2^{-m}$ from the condition on the diameters of $U_m(a)$ and $U_m(b)$. Hence we have $ay<2^{1-m}<r$, and therefore y is in $S(a)$, which is contained in $U_n(a)$. Hence $U_m(b) \subset U_n(a)$.

The condition of this theorem has an advantage over the condition of Alexandroff and Urysohn. The latter condition postulates the existence of families of covering sets $\{G_n\}$ having certain properties, and in terms of these sets the distance function is defined. Given a neighborhood space, it might be difficult to determine whether such families of sets $\{G_n\}$ could be found. The condition of the present theorem also leads to the existence of such sets with the additional information that they are to be found among the original neighborhoods of the space.

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AN INVOLUTORIAL LINE TRANSFORMATION DETERMINED BY A CONGRUENCE OF TWISTED CUBIC CURVES*

BY J. M. CLARKSON

1. *Introduction.* Consider the pencils of quadric cones $K_1-\alpha K_2=0$, $L_1-\beta L_2=0$, each pencil having a common vertex which lies on all of the cones of the other pencil. For a given α, β the curve $C(\alpha, \beta)$ of intersection of the cones is composite, consisting of the line l of the vertices of the two pencils and a twisted cubic curve $C_3(\alpha, \beta)$. As α, β take on all values independently, $C_3(\alpha, \beta)$ describes a congruence of space cubic curves. An arbitrary line t of space will be bisecant to just one $C_3(\alpha, \beta)$, for any three points of space will determine a set of values for the parameters α, β, ρ in the system

$$(1) \qquad (K_1 - \alpha K_2) - \rho(L_1 - \beta L_2) = 0$$

of quadric surfaces, and if these three points be chosen on t, then t lies on the quadric of (1) so determined and will meet $C_3(\alpha, \beta)$ twice. We shall henceforth write $C_3(t)$ for this curve.

Now consider a fixed plane π and in this plane a Cremona involutorial transformation Γ of order n having a curve Δ_m of

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order m as its curve of invariant points. In case $n=2$, and Γ has only a finite number of invariant points, then $m=0$. The line t meets π in one point T whose image in Γ is T'. Through T' there exists one and only one bisecant t' of $C_3(t)$. We shall say that t is transformed into t' by the line transformation I and write $t \sim (I)t'$. This line transformation is involutorial, since t' determines the same $C_3(t)$ and T' has for its image in Γ the point T.

2. *The Order of I.* Let the Plücker coordinates of t be $(y_1, y_2, y_3, y_4, y_5, y_6)$. The three points chosen on t have coordinates which are linear functions of y_i, and the values of α, β are of the form $u_1(y)/u_2(y)$, $v_1(y)/v_2(y)$, where u_i, v_i are found to be functions of degree one in y_j.* Hence

$$
\begin{aligned}
u_2(y)K_1(z) - u_1(y)K_2(z) &= 0,\\
v_2(y)L_1(z) - v_1(y)L_2(z) &= 0,
\end{aligned}
\tag{2}
$$

are the equations of $C_3(t)$, where z_i are the running coordinates of points in space.

Now the quadric of the pencil

$$
[u_2(y)K_1(z) - u_1(y)K_2(z)] - \lambda[v_2(y)L_1(z) - v_1(y)L_2(z)] = 0 \tag{3}
$$

which passes through T' passes through $C_3(t)$. The coordinates of T' are functions of degree n in y_i, and hence the coefficients in (3) are of degree $(2n+1)$ in y_i. Through T' passes one generator of each regulus of (3), but all of the generators of one regulus meet $C_3(t)$ twice and those of the other regulus meet it only once, and meet the line l which lies on all quadrics of the pencil. Hence t' is that generator of (3) which passes through T' and belongs to the same regulus to which l belongs. The Plücker coordinates x_j of t' are functions of degree 2 in the coefficients of the equation (3) and are thus functions of degree $(4n+2)$ in y_i. Hence I is of order $(4n+2)$.

3. *The Invariant Lines of I.* When t meets Δ_m, the curve of invariant points of Γ, then $T' \equiv T$, and hence $t' \equiv t$. Thus the invariant lines of I form a special complex of order m, having Δ_m as directrix. In the case of the quadratic involution for Γ,

* The process used is that given in a paper by Snyder and Clarkson in this Bulletin, vol. 40 (1934), pp. 441–448.

having only a finite number of invariant points, this configuration of invariant lines becomes a finite number of bundles, having vertices at the invariant points. If however, $m>0$, any line t_π in the plane π meets Δ_m, and t_π is invariant and is counted m times in the complex. Thus the plane π is a singular plane for the complex. We shall see further that t_π is also singular in I.

4. *The Singular Lines of I.* When $t\equiv l$, then $C_3(t)$ is indeterminate, since l meets every $C_3(\alpha, \beta)$ twice. But the point L where l meets π under Γ an image L', and since I is involutorial, any line of the bundle (L') may be considered as the conjugate of l.

When t meets π in a fundamental point R of Γ, then t' is any generator of a ruled surface Φ_{4r} of order 4 times the multiplicity r of R in Γ. In order to see this, consider the plane curve ϕ_r in π, the principal curve corresponding to R in Γ. Each point of ϕ_r may be taken as R' and through this point passes one bisecant of $C_3(t)$. Thus ϕ_r is simple on Φ_{4r}. But, in π lie three bisecants of $C_3(t)$ and each of these meets ϕ_r in r points, thus making each bisecant of multiplicity r on Φ_{4r}. The plane π then meets Φ_{4r} in ϕ_r and in three lines each of multiplicity r, making a total intersection of order $4r$.

When $t\equiv t_\pi$, any line in π, then any point of τ_n, the image of t under Γ, may be taken as T' and thus t_π is transformed by I into a ruled surface of order $4n$.

5. *The Special Linear Complex with Axis l.* We have already seen that the conjugate t' of an arbitrary line t does not meet the line l of the vertices of the pencils of cones, and since I is involutorial, a line t belonging to the special linear complex with axis l must have for its conjugate a line t' which also belongs to this special complex. The Plücker coordinates of t will be such that if t belongs to the regulus of (1) to which l does not belong, so will the Plücker coordinates of t' be such that t' belongs to this regulus also. Thus the special linear complex with axis l is invariant as a whole, but not line by line.

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ERRATUM

On page 877 of the December, 1936, issue of this Bulletin, in line 16, change $O(n^{-i})$, to $o(n^{-i})$.