exp (nW), n any integer. Then N is a discrete subgroup of the central of G, and so (see [1], p. 12) G/N is a Lie group locally topologically isomorphic with G.

But the homomorphism $G\to G/N$ carries S_3 into S_3/N, which is simply isomorphic with $G_3/N=G_3^*$. This and the corollary to Theorem 1 complete the proof.

E. Cartan [5] has shown that the universal covering group of the group of projective transformations of the line is topologically isomorphic in the large with no linear group.

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CHARACTERISTICS OF BIRATIONAL TRANSFORMS IN S_r

BY B. C. WONG

1. *Introduction*. Consider a k-dimensional variety, V_k^n, of order n in an r-space, S_r. Let us project V_k^n from a general $(r-k-t-1)$-space of S_r upon a general $(k+t)$-space of S_r and denote the projection by ${}_tV_k^n$. We are supposing that $1\leqq t\leqq k$. Then upon ${}_tV_k^n$ lies a double variety, D_{k-t}, of dimension $k-t$ and order b_t and upon D_{k-t} lies a pinch variety, W_{k-t-1}, of dimension $k-t-1$ and order j_{t+1}. Since the symbol W_{-1} is without meaning, we thus obtain $2k-1$ characteristics $b_1, b_2, \cdots, b_k, j_2, j_3, \cdots, j_k$. The symbol j_1 has a meaning which will be explained subsequently.

Now let a general $(r-k+q-2)$-space, $S_{r-k+q-2}$, $(1\leqq q\leqq k)$, be given in S_r. Through this $S_{r-k+q-2}$ pass ∞^{k-q+1} primes of S_r and ∞^{k-q} of these are tangent to V_k^n. The points of contact form a $(k-q)$-dimensional variety, U_{k-q}. Denote its order by m_q. Thus we obtain k further characteristics $m_1, m_2, \cdots, m_k$. If we project V_k^n upon a $(k+1)$-space, S_{k+1}, of S_r, we see that m_q is the class of the V_q^n in which a $(q+1)$-space of S_{k+1} meets the projected variety. We also say that m_q is the class of the q-dimensional variety in which an $(r-k+q)$-space of S_r meets V_k^n.

In the case where V_k^n is the complete intersection of $r-k$ general primals, of orders $n_1, n_2, \cdots, n_{r-k}$, respectively, in S_r, the values of j_t, b_t, m_q are known* and they are

$$\text{(I)} \qquad j_t = n_1 n_2 \cdots n_{r-k} \sum (n_1 - 1)(n_2 - 1) \cdots (n_t - 1),$$

$$\begin{aligned}\text{(II)} \qquad b_t &= \frac{1}{2} n_1 n_2 \cdots n_{r-k}\big[n_1 n_2 \cdots n_{r-k} - 1 - \sum (n_1 - 1) \\ &\quad - \sum (n_1 - 1)(n_2 - 1) - \cdots \\ &\quad - \sum (n_1 - 1)(n_2 - 1) \cdots (n_t - 1)\big] \\ &= \frac{1}{2} n_1 n_2 \cdots n_{r-k}\big[\sum (n_1 - 1)(n_2 - 1) \cdots (n_{t+1} - 1) \\ &\quad + \sum (n_1 - 1)(n_2 - 1) \cdots (n_{t+2} - 1) + \cdots \\ &\quad + \sum (n_1 - 1)(n_2 - 1) \cdots (n_{r-k} - 1)\big],\end{aligned}$$

$$\text{(III)} \qquad m_q = n_1 n_2 \cdots n_{r-k} \sum_h \sum (n_1 - 1)^{h_1} (n_2 - 1)^{h_2} \cdots (n_q - 1)^{h_q},$$

where

$$h = h_1 + h_2 + \cdots + h_q = q.$$

We shall refer to these values later.

In this paper we propose to determine the values of the same characteristics for the variety V_k^n in S_r which we consider as the birational transform of a k-dimensional variety, say Φ_k^ν, of order ν in a ρ-space, Σ_ρ, for $\rho < r$. We confine ourselves to the case where Φ_k^ν is the complete intersection of $\rho - k$ general primals of Σ_ρ, of respective orders $\nu_1, \nu_2, \cdots, \nu_{\rho-k}$, given by the equations

$$(1) \qquad F^{(1)} = 0, \ F^{(2)} = 0, \ \cdots, \ F^{(\rho-k)} = 0,$$

* C. Segre, *Mehrdimensionale Räume*, Encyklopädie der Mathematischen Wissenschaften, III C7, pp. 944–945. Also B. C. Wong, *On certain characteristics of k-dimensional varieties in r-space*, this Bulletin, vol. 38 (1932), pp. 725–730.

where $F^{(i)}$ is a homogeneous function of degree ν_i in the variables $\xi_0, \xi_1, \cdots, \xi_\rho$. The order of Φ_k^ν is $\nu=\nu_1\nu_2\cdots\nu_{\rho-k}$. The corresponding characteristics η_t, β_t, μ_q of Φ_k^ν are given by (I), (II), (III) if we replace in the right-hand members r by ρ and n_i by ν_i.

We suppose that the transformation of Φ_k^ν into V_k^n is accomplished by means of a general linear ∞^r-system, $|\psi|$, without base varieties of any kind, of $(k-1)$-dimensional varieties of order νN, and that $|\psi|$ is the intersection of Φ_k^ν and a general linear ∞^r-system, $|\phi|$, of primals of order N, none passing through Φ_k^ν, given by the equation

$$(2) \qquad a_0\phi^{(0)} + a_1\phi^{(1)} + \cdots + a_r\phi^{(r)} = 0,$$

the ϕ's being linearly independent homogeneous polynomials of degree N in the $(\rho+1)$ ξ's. Then, the order of V_k^n is $n=\nu N^k=\nu_1\nu_2\cdots\nu_{\rho-k}N^k$. The coordinates of the points on V_k^n are given by

$$\sigma x_0 = \phi^{(0)}(\xi_0, \xi_1, \cdots, \xi_\rho), \qquad \sigma x_1 = \phi^{(1)}(\xi_0, \xi_1, \cdots, \xi_\rho), \\ \cdots, \qquad \sigma x_r = \phi^{(r)}(\xi_0, \xi_1, \cdots, \xi_\rho),$$

where the ξ's satisfy equations (1).

It is to be noted that an h-dimensional locus of order l on Φ_k^ν goes into an h-dimensional locus of order lN^h on V_k^n. For $h=k$, Φ_k^ν goes into V_k^n, where $n=\nu N^k$.

We shall first, in §2, derive a general relation connecting the b's and the j's for a general variety which has no extraordinary singular points. The determination of the values of the m's of our variety V_k^n will be given in §3 and the determination of those of the j's in §4. The values of the b's will then be obtained with the aid of the relation derived in §2. Incidentally, we find it interesting to express the m's and j's in terms of the μ's and η's, respectively, of Φ_k^ν.

2. *The Relation between the b's and the j's.* Let C^l be a curve of order l, in a space of dimension greater than 2, whose points are paired in an (irrational) involution I_2. Suppose that C^l has d actual nodes at each of which two corresponding points of I_2 coincide but lie on different branches of the curve. If i denotes the number of simple points of C^l at each of which two corresponding points of I_2 become united, the order of the ruled sur-

face, which may be a cone, whose generators are lines joining corresponding points of the involution is, as is well known,

$$R=(2l-i-2d)/2.$$

Now consider a general k-dimensional variety V_k, of any order n, without extraordinary singular points, in S_r and let it be intersected by a general $(r-k+t)$-space of S_r in a V_t. If we project V_t upon a $(2t-1)$-space, S_{2t-1}, we see that the projection ${}_{t-1}V_t$ has a double curve D_1 of order b_{t-1} and j_t pinch points. This ${}_{t-1}V_t$ may certainly be regarded as the projection of a ${}_tV_t$ in a $(2t)$-space, S_{2t}, the ${}_tV_t$ being assumed to be a projection of V_t. Let Z be the point, taken in a general position of S_{2t}, from which ${}_tV_t$ is projected into ${}_{t-1}V_t$ in S_{2t-1}. There are ∞^1 lines through Z meeting ${}_tV_t$ in two distinct points and the locus of these lines is a ruled surface, in fact a cone, of order b_{t-1}. This cone meets ${}_tV_t$ in a curve c of order $2b_{t-1}$, of which the double curve D_1 on ${}_{t-1}V_t$ is the projection. The curve c has b_t actual nodes which are the improper double points of ${}_tV_t$. There are j_t elements of the cone tangent to ${}_tV_t$ and also to c. The projections of the points of contact are the pinch points on ${}_{t-1}V_t$. Now on c is an involution of pairs of points set up by the elements of the cone. There are b_t points each of which is the union of two corresponding points on different branches of the curve and j_t points each of which is the union of two corresponding points on a simple branch of the curve. Putting $R=b_{t-1}$, $l=2b_{t-1}$, $i=j_t$, $d=b_t$ in the relation of the paragraph just preceding, we have the desired relation $b_{t-1}=(4b_{t-1}-j_t-2b_t)/2$, or

$$2b_{t-2} = j_t + 2b_t. \tag{3}$$

By letting $t=1, 2, \cdots, k$, successively, we obtain

$$2b_0 = j_1 + 2b_1,\ 2b_1 = j_2 + 2b_2, \cdots, \quad 2b_{k-1} = j_k + 2b_k.$$

As we shall see presently, $b_0=n(n-1)/2$ and j_1 is identical with the class m_1 of a plane section of the projection ${}_1V_k$ in a $(k+1)$-space. The relation (3) may be replaced by the relation

$$2b_s - 2b_t = j_{s+1} + j_{s+2} + \cdots + j_t, \qquad (s > t). \tag{4}$$

Note that this relation, or relation (3), is satisfied by (I) and (II).

3. *The Determination of the m's.* Returning to the V_k^n which is the birational transform of Φ_k^ν in Σ_ρ, we see at once that m_q, $(1 \leqq q \leqq k)$, is the order of the variety U_{k-q} on V_k^n which has for image on Φ_k^ν the complete intersection Θ_{k-q} of Φ_k^ν and the Jacobian variety $\omega_{\rho-q}$ of the $\rho-k$ primals, given by (1), intersecting in Φ_k^ν, and any $k-q+2$ independent primals of the system $|\phi|$, say $\phi^{(1)}=0, \phi^{(2)}=0, \cdots, \phi^{(k-q+2)}=0$. Θ_{k-q} is the locus of the points of contact between Φ_k^ν and the ∞^{k-q} primals of the ∞^{k-q+1}-system determined by the $k-q+2$ primals just mentioned which are tangent to Φ_k^ν. The conditions of contact are given by

$$\left\| \begin{matrix} F_0^{(1)} & F_1^{(1)} & \cdots F_\rho^{(1)} \\ \cdot & \cdot & \cdots \cdot \\ F_0^{(\rho-k)} & F_1^{(\rho-k)} & \cdots F_\rho^{(\rho-k)} \\ \phi_0^{(1)} & \phi_1^{(1)} & \cdots \phi_\rho^{(1)} \\ \cdot & \cdot & \cdots \cdot \\ \phi_0^{(k-q+2)} & \phi_1^{(k-q+2)} & \cdots \phi_\rho^{(k-q+2)} \end{matrix} \right\| = 0,$$

where $F_i^{(h)}$, $\phi_i^{(h)}$ are written in place of $\partial F^{(h)}/\partial \xi_i$, $\partial \phi^{(h)}/\partial \xi_i$, respectively. This equality represents the Jacobian $\omega_{\rho-q}$. The matrix being of $\rho+1$ columns and $\rho-q+2$ rows, the order of $\omega_{\rho-q}$ is*

$$\begin{aligned} H_q = \sum_{i=0}^{q} \binom{k+1-i}{q-i} (N-1)^{q-i} \\ \cdot \sum_i \sum (\nu_1-1)^{h_1}(\nu_2-1)^{h_2} \cdots (\nu_i-1)^{h_i}, \end{aligned} \tag{5}$$

where

$$h_1 + h_2 + \cdots + h_i = i.$$

Then the order of Θ_{k-q} is $\nu_1\nu_2 \cdots \nu_{\rho-k}H_q$ and therefore the order of U_{k-q} on V_k^n which is the transform of Θ_{k-q} is

$$m_q = \nu_1\nu_2 \cdots \nu_{\rho-k}N^{k-q}H_q. \tag{6}$$

It is of interest to express m_q in terms of the μ's belonging to Φ_k^ν. The various values of the μ's are obtained from (III) by changing r to ρ and n_i to ν_i. By taking account of (5), we have, from (6), writing μ_0 in place of ν or $\nu_1\nu_2 \cdots \nu_{\rho-k}$,

* Salmon, *Modern Higher Algebra*, 4th ed., Lesson 19.

$$m_q = N^{k-q}\sum_{i=0}^{q}\binom{k+1-i}{q-i}(N-1)^{q-i}\mu_i$$
$$= N^{k-q}\left[\binom{k+1}{q}(N-1)^q\mu_0 + \binom{k}{q-1}(N-1)^{q-1}\mu_1 + \cdots + \mu_q\right].$$

4. *The Determination of the j's and the b's.* The projection ${}_{t-1}V_k^n$, in a $(k+t-1)$-space of S_r, of V_k^n has a pinch variety W_{k-t} of order j_t. This W_{k-t} has for image on Φ_k^ν the complete intersection of Φ_k^ν and the variety $\pi_{\rho-t}$ common to all the primals each of which is the Jacobian of $\rho+1$ of the following $\rho+t$ primals:

$$F^{(1)}=0,\qquad F^{(2)}=0,\qquad \cdots,\qquad F^{(\rho-k)}=0,$$
$$\phi^{(1)}=0,\qquad \phi^{(2)}=0,\qquad \cdots,\qquad \phi^{(k+t)}=0.$$

The $k+t$ primals represented by the ϕ's equated to zero are supposed to be independent members of the system $|\phi|$ given by (2). The equations of $\pi_{\rho-t}$ are

$$\left\|\begin{matrix} F_0^{(1)} & F_1^{(1)} & \cdots F_\rho^{(1)} \\ \cdot & \cdot & \cdot\quad \cdot \\ F_0^{(\rho-k)} & F_1^{(\rho-k)} & \cdots F_\rho^{(\rho-k)} \\ \phi_0^{(1)} & \phi_1^{(1)} & \cdots \phi_\rho^{(1)} \\ \cdot & \cdot & \cdot\quad \cdot \\ \phi_0^{(k+t)} & \phi_1^{(k+t)} & \cdots \phi_\rho^{(k+t)} \end{matrix}\right\| = 0.$$

The left-hand member being a matrix of $\rho+t$ rows and $\rho+1$ columns, the order of $\pi_{\rho-t}$ is*

$$(7)\quad \begin{aligned} C_t &= \sum_{i=0}^{t}\binom{k+t}{i}(N-1)^i\sum(\nu_1-1)(\nu_2-1)\cdots(\nu_{t-i}-1) \\ &= \sum_{i=0}^{t}\binom{k+t}{t-i}(N-1)^{t-i}\sum(\nu_1-1)(\nu_2-1)\cdots(\nu_i-1), \end{aligned}$$

* Salmon, loc. cit.

and the order of the intersection of Φ_k^ν and $\pi_{\rho-t}$ is νC_t. Therefore, the order of W_{k-t} is

$$
\begin{aligned}
j_t &= \nu N^{k-t} C_t \\
&= \nu_1\nu_2 \cdots \nu_{\rho-k} N^{k-t} \sum_{i=0}^{t} \binom{k+t}{t-i} (N-1)^{t-i} \\
&\qquad\qquad \cdot \sum (\nu_1 - 1)(\nu_2 - 1) \cdots (\nu_i - 1).
\end{aligned}
\tag{8}
$$

Thus, for $t=1$, we have

$$
j_1 = \nu_1\nu_2 \cdots \nu_{\rho-k} N^{k-1}\left[(k+1)(N-1) + \sum (\nu_1 - 1)\right] = m_1.
$$

Now in terms of the η's of Φ_k^ν we have, from (I) and (8),

$$
\begin{aligned}
j_t &= N^{k-t} \sum_{i=0}^{t} \binom{k+t}{t-i} (N-1)^{t-i}\eta_i \\
&= N^{k-t}\left[\binom{k+t}{t}(N-1)^t\eta_0 + \binom{k+t}{t-1}(N-1)^{t-1}\eta_1 + \cdots \right. \\
&\qquad\qquad \left. + \binom{k+t}{1}(N-1)\eta_{t-1} + \eta_t\right],
\end{aligned}
$$

where $\eta_0 = \nu = \nu_1\nu_2 \cdots \nu_{\rho-k}$.

To determine the b's we make use of the relation (3) or (4) of §2. A little calculation yields

$$
b_t = \frac{1}{2}\nu\left[\nu N^{2k} - \sum_{h=0}^{t} N^{k-h} C_h\right],
$$

where C_h is given by (7) if we replace in it t by h. Since any $(k-t)$-dimensional variety of order l on Φ_k^ν goes into a $(k-t)$-dimensional variety of order lN^{k-t} on V_k^n, we see that

$$
\frac{2b_t}{N^{k-t}} = \nu^2 N^{k+t} - \nu \sum_{h=0}^{t} N^{t-h} C_h
$$

is the order of the image on Φ_k^ν of the double variety D_{k-t} on the projection ${}_tV_k^n$ in a $(k+t)$-space.

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