4. *Remark*. Let the sequence $E_1, E_2, \cdots$ be as in §2. Then we can even assert *that for every* $\lambda<1$ *there exists an infinite subsequence* $E_{i_1}, E_{i_2}, \cdots$ *such that for every* $p$ *and* $q$

$$\mu(E_{i_p}E_{i_q}) \geqq \lambda m^2.$$

We show first that there exists an infinite subsequence $E_{k_1}, E_{k_2}, \cdots$ such that $\mu(E_{k_1}, E_{k_p}) \geqq \lambda m^2$ for every $p$. Suppose that no such subsequence exists; then to every $n=1, 2, \cdots$ belongs a $p_n$ such that

$$\mu(E_nE_m) < \lambda m^2 \quad \text{for} \quad m \geqq n + p_n.$$

Writing $n_1=1$, $n_2=n_1+p_{n_1}$, $n_3=n_2+p_{n_2}, \cdots$, we have then for every $i$ and $k$,

$$\mu(E_{n_i}E_{n_k}) < \lambda m^2,$$

which contradicts the theorem of §2. The proof is now easily completed by applying the diagonal principle.

CAMBRIDGE, MASSACHUSETTS

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# ON THE ZEROS OF THE DERIVATIVE OF A RATIONAL FUNCTION*

BY MORRIS MARDEN

1. *Introduction*. The primary object of this note is to give a simple solution of a problem already discussed by many authors including the present one.† It is the problem of determining the regions within which lie the zeros of the derivative of a rational function when the zeros and poles of the function lie in prescribed circular regions.

THEOREM 1.‡ *For* $j=0, 1, \cdots, p$ *let* $r_j$ *and* $\sigma_j$ *be real constants*

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† For an expository account and list of references see M. Marden, American Mathematical Monthly, vol. 42 (1935), pp. 277–286, hereafter referred to as Marden I.

‡ See M. Marden, Transactions of this Society, vol. 32 (1930), pp. 81–109, hereafter referred to as Marden II.

*with* $\sigma_j^2=1$; *let* $Z_j$ *denote the circular region defined by the inequality*

$$\sigma_j Z_j(z) \equiv \sigma_j(|z-\alpha_j|^2 - r_j^2) \leqq 0,$$

*and let* $z_j$ *be an arbitrary point of the region* $Z_j$. *Then every zero of the derivative of the function**

$$f(z) = \prod_{j=0}^{p} (z-z_j)^{m_j}$$

*satisfies at least one of the* $p+2$ *inequalities*

$$(1) \qquad \sigma_j Z_j(z) \leqq 0, \qquad (j=0,1,\cdots,p),$$

$$(2) \qquad \frac{Z(z)}{\prod_{j=0}^{j=p} Z_j(z)} \equiv \left| \sum_{j=0}^{p} \frac{m_j(\bar{\alpha}_j - \bar{z})}{Z_j(z)} \right|^2 - \left( \sum_{j=0}^{p} \frac{|m_j|\sigma_j r_j}{Z_j(z)} \right)^2 \leqq 0.$$

Theorem 1 holds even when the $m_j$ are complex numbers. When, however, they are all real, the use of the identities

$$(\alpha_j - z)(\bar{\alpha}_k - \bar{z}) + (\bar{\alpha}_j - \bar{z})(\alpha_k - z) = |\alpha_j - z|^2 + |\alpha_k - z|^2 - |\alpha_j - \alpha_k|^2,$$

and

$$|\alpha_j - z|^2 = Z_j(z) + r_j^2,$$

enables one, after expanding (2), to write

$$(3) \qquad \frac{Z(z)}{\prod_{j=0}^{j=p} Z_j(z)} \equiv \sum_{j=0}^{p} \frac{nm_j}{Z_j(z)} - \sum_{j=0}^{p} \sum_{k=j+1}^{p} \frac{m_j m_k \tau_{jk}}{Z_j(z) Z_k(z)},$$

where $n=\sum m_j$ and

$$\tau_{jk} = |\alpha_j - \alpha_k|^2 - (|m_j| m_j^{-1}\sigma_j r_j - |m_k| m_k^{-1}\sigma_k r_k)^2.$$

The latter is the square of the common external or internal tangent of the circles $Z_j(z)=0$ and $Z_k(z)=0$ according as the product $|m_j m_k|(m_j m_k)^{-1}\sigma_j\sigma_k$ is positive or negative. The equation $Z(z)=0$ represents for $n\neq 0$ a $p$-circular $2p$-ic and for $n=0$ in general a $(p-1)$-circular $2(p-1)$-ic curve. The properties of these curves are studied in Marden II, p. 92.

* Where no limits are indicated, a product or summation is to be taken from $j=0$ to $j=p$ and from $k=j+1$ to $k=p$.

2. *Three Lemmas.** (I). *If the points $t_0, t_1, \cdots, t_p$ varying independently of one another describe the closed interiors of the circles $T_0, T_1, \cdots, T_p$, respectively, the center and radius of $T_j$ being $\gamma_j$ and $\rho_j$, then the point $w=\sum_{j=0}^{p} m_j t_j$ describes the closed interior of a circle $W$ with center at $\gamma=\sum_{j=0}^{p} m_j\gamma_j$ and radius of $\rho=\sum_{j=0}^{p}|m_j|\rho_j$.*

For

$$|w-\gamma| = \left|\sum_{j=0}^{p} m_j(t_j-\gamma_j)\right| \leqq \rho;$$

conversely, if $k$ and $\theta$ are arbitrary, $0\leqq k\leqq 1$, and if $m_j(t_j-\gamma_j) = k|m_j|\rho_j e^{i\theta}$, then $w-\gamma=\sum k|m_j|\rho_j e^{i\theta}=k\rho e^{i\theta}$.

(II). *If the points $t_j$, $(j\geqq 1)$, vary as in Lemma* (I), *but the point $t_0$ describes the closed exterior of its circle $T_0$, the locus of the point $w$ is the closed exterior of a circle with center at $\gamma=\sum_{j=0}^{p} m_j\gamma_j$ and radius $\rho=2|m_0|\rho_0-\sum_{j=0}^{p}|m_j|\rho_j$ provided $\rho>0$, and is the entire plane if $\rho\leqq 0$.*

For, when $\rho>0$,

$$|w-\gamma| \geqq |m_0(t_0-\gamma_0)| - \left|\sum_{j=1}^{p} m_j(t_j-\gamma_j)\right| \geqq \rho;$$

conversely, if $k$ and $\theta$ are arbitrary with $k\geqq 1$, and if

$$m_0(t_0-\gamma_0) = [\,|m_0|\rho_0+(k-1)\rho]e^{i\theta},$$

and

$$m_j(t_j-\gamma_j) = -|m_j|\rho_j e^{i\theta}, \qquad (j\geqq 1),$$

then $w-\gamma=k\rho e^{i\theta}$.

If $\rho_0$ decreases while $\rho_j$, $(j\geqq 1)$, remain constant, $\rho$ will approach zero and the locus of $w$ will become the entire plane. The locus is, therefore, the entire plane for $\rho\leqq 0$.

(III). *If the points $t_j$, $(j>k\geqq 1)$, vary as in Lemma* (I), *but the points $t_j$, $(j\leqq k)$, describe the exteriors of their circles $T_j$, the locus $W$ of $w$ is the entire plane.*

For, if each $t_j$, $(1\leqq j\leqq k)$, were to vary merely interior to a

* See J. L. Walsh, Transactions of this Society, vol. 24 (1922), p. 61 and p. 169; also H. Minkowski, Collected Works, vol. 2, p. 177.

circle $T_j'$ drawn exterior to but not enclosing the circle $T_j$ while the remaining $t_j$ vary as indicated in Lemma (III), and if the radius $\rho_j'$ of $T_j'$ were chosen so that

$$| m_0 | \rho_0 - \sum_{j=1}^{k} | m_j | \rho_j' - \sum_{j=k+1}^{p} | m_j | \rho_j = 0,$$

then by Lemma (II) the locus of $w$ would be the entire plane.

3. *Proof of Theorem* 1. Let $z$ be any fixed point exterior to all the regions $Z_j$; that is, let $z$ be such that $\sigma_j Z_j(z) > 0$, all $j$. Since $\sigma_j Z_j(z_j) \leqq 0$, point $t_j = (z_j - z)^{-1}$ lies in or on the circle $T_j$ with center $\gamma_j = (\bar{\alpha}_j - \bar{z})/Z_j(z)$ and radius $\rho_j = (\sigma_j \gamma_j)/Z_j(z)$. According to Lemma (I), the locus $W_z$ of the point $w = \sum m_j t_j$ will be defined by the inequality:

$$(4) \qquad \left| w - \sum_{j=0}^{p} \frac{m_j(\bar{\alpha}_j - \bar{z})}{Z_j(z)} \right|^2 - \left( \sum_{j=0}^{p} \frac{| m_j | \sigma_j r_j}{Z_j(z)} \right)^2 \leqq 0.$$

Now, in order to be a zero of $f'(z)$, point $z$ must be a root of the equation

$$(5) \qquad - \frac{f'(z)}{f(z)} = \sum_{j=0}^{p} \frac{m_j}{z_j - z} = 0;$$

that is, point $w = 0$ must satisfy inequality (4). Hence, any zero of $f'(z)$, not satisfying any of the inequalities (1), must satisfy (2); that is to say, $\sigma Z(z) \leqq 0$, where $\sigma = \prod \sigma_j$.

4. *A Locus Problem*. What then is the locus $Z$ of the zeros of $f'(z)/f(z)$ when the points $z_j$ vary independently within their circular regions $Z_j$?

Theorem 1 reveals that $\sigma Z(z) \leqq 0$ for any point $z$ of locus $Z$ exterior to all the regions $Z_j$. Conversely, if exterior to all the regions $Z_j$, any point $z$ for which $\sigma Z(z) \leqq 0$ belongs to the locus $Z$. With the aid of Lemma (II), it can be shown that a point $z$ interior to just one region $Z_j$ belongs to locus $Z$ if and only if either $\sigma Z(z) \leqq 0$ or $\sigma S(z) \leqq 0$, where

$$\frac{S(z)}{\prod Z_j(z)} = \sum_{j=0}^{p} \frac{| m_j | r_j \sigma_j}{Z_j(z)}.$$

(The curve $S(z) = 0$, in general a $p$-circular $2p$-ic, consists only of points on the boundaries of two or more regions $Z_j$ or interior to at least one region $Z_j$, the points $z$ interior to just one region $Z_j$ satisfying inequality $\sigma Z(z) \leqq 0$.) Likewise, with the aid of Lemma (III), it can be shown that every point common to two or more regions $Z_j$ belongs to locus $Z$.

If a given point $z$ is to be on the boundary of locus $Z$, point $w=0$ must cease to be a point of $W_z$ whenever the regions $Z_j$ and hence $W_z$ are diminished, no matter how slightly. That is to say, the point $w=0$ must be on the boundary of the locus $W_z$. This implies that $Z(z)=0$ if $z$, a boundary point of locus $Z$, is exterior to all the regions $Z_j$ or interior to just one region $Z_j$ with $\sigma Z(z) \leqq 0$. It also implies that no boundary point $z$ of locus $Z$ may be either interior to just one region $Z_j$ with $\sigma S(z) \leqq 0$, or interior to two or more regions $Z_j$; for, in those cases, the locus $W_z$ is the entire plane.

In short, the locus Z is a set of regions bounded by the ovals of the curve $Z(z)=0$, each region, according to Marden II, being simply-connected.

5. *Applications*. When $p=2$ and $n=m_0+m_1+m_2=0$, equation (5) may be written as the cross-ratio

$$(z_0z_1z_2z) \equiv \frac{(z_0 - z_2)(z_1 - z)}{(z_0 - z)(z_1 - z_2)} = - \frac{m_1}{m_0}. \tag{6}$$

Here

$$Z(z) \equiv - m_0m_1r_{01}Z_2(z) - m_1m_2r_{12}Z_0(z) - m_2m_0r_{20}Z_1(z) = 0,$$

$$S(z) \equiv | m_0 | r_0\sigma_0Z_1(z)Z_2(z) + | m_1 | r_1\sigma_1Z_2(z)Z_0(z) + | m_2 | r_2\sigma_2Z_0(z)Z_1(z) = 0,$$

represent in general a circle and bicircular quartic, respectively. If $\lambda$ denotes the coefficient of the term $(x^2+y^2)$ in the expression $Z(z)$, the region $\sigma Z(z) \leqq 0$ is the interior or exterior of the circle $Z(z)=0$ according as $\sigma\lambda>0$ or $\sigma\lambda<0$. Hence, if all the points for which $\sigma S(z) \leqq 0$ lie in the region $\sigma Z(z) \leqq 0$, the locus $Z$ will be the interior or exterior of circle $Z(z)=0$ according as $\sigma\lambda>0$ or $\sigma\lambda<0$. If, however, not all the points for which $\sigma S(z) \leqq 0$ lie in the region $\sigma Z(z) \leqq 0$, the locus will be the entire plane. This discussion verifies the following theorem due to Walsh.*

* J. L. Walsh, Transactions of this Society, vol. 22 (1921), pp. 101–116, and Rendiconti di Palermo, vol. 46 (1922), pp. 1–13. See also A. B. Coble, this Bulletin, vol. 27 (1921), pp. 434–437; T. Nakahara, Tôhoku Mathematical Journal, vol. 23 (1924), p. 97; and Marden II.

*If the points $z_0$, $z_1$, and $z_2$ varying independently of one another describe given circular regions $Z_0$, $Z_1$, and $Z_2$, then the point $z$ defined by the constant cross-ratio $(z_0z_1z_2z)=c$ also describes a circular region $Z$.*

On allowing a number of the regions $Z_j$ to coincide, we deduce from Theorem 1 the following corollary.

COROLLARY 1. *If all the zeros of a polynomial $f_j(z)$ of degree $n_j$ lie in the circular region $Z_j$, then every zero of the derivative of the product*

$$\prod_{j=0}^{p} [f_j(z)]^{q_j}$$

*will satisfy at least one of the $p+2$ inequalities* (1) *and* (2) *with $m_j=n_jq_j$.**

In particular, upon setting $f_j(z)=f(z)-\gamma_j$, we obtain from Corollary 1 a generalization of a theorem stated by Jentsch and proved by Fekete.†

COROLLARY 2. *If all the points at which a given polynomial $f(z)$ takes on the value $\gamma_j$ lie in the circular region $Z_j$, then every root $z$ of the equation*

$$\sum_{j=0}^{p} \frac{m_j}{f(z)-\gamma_j} = 0$$

*satisfies at least one of the $p+2$ inequalities* (1) *and* (2).

6. *Generalizations.* By requiring $w=\lambda$ instead of $w=0$ to satisfy inequality (4), we are led to the following result.

THEOREM 2. *Under the hypotheses of Theorem* 1 *or of Corollary* 1, *every zero of the function $f'(z)+\lambda f(z)$ satisfies at least one of the $p+2$ inequalities*

$$\sigma_j Z_j(z) \leqq 0, \qquad (j=0,1,\cdots,p), \tag{7}$$

$$\left|\lambda-\sum_{j=0}^{p}\frac{m_j(\bar{\alpha}_j-\bar{z})}{Z_j(z)}\right|^2-\left(\sum_{j=0}^{p}\frac{|m_j|\,\sigma_j r_j}{Z_j(z)}\right)^2 \leqq 0. \tag{8}$$

* This corollary contains as special cases a number of important theorems due to Gauss-Lucas, Laguerre, Bôcher, and Walsh. See Marden I.

† R. Jentsch, Archiv der Mathematik und Physik, vol. 25 (1917), p. 196, prob. 526; M. Fekete, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 31 (1922), pp. 42–48; Pólya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, p. 61, probs. 126–127.

If all the $m_j$ are real, the left-hand side of (8) may be rewritten, with the aid of the identities given in §1, as

$$\sum_{j=0}^{p} \frac{|\lambda|^2 m_j \Gamma_j(z)}{n Z_j(z)} - \sum_{j=0}^{p} \sum_{k=j+1}^{p} \frac{m_j m_k \tau_{jk}}{Z_j(z) Z_k(z)}, \tag{9}$$

where

$$\Gamma_j(z) \equiv |z - (\alpha - n\lambda^{-1})|^2 - r_j^2 .$$

The equation $\Gamma_j(z) = 0$ represents the circle obtained by translating the circle $Z_j(z) = 0$ in the direction and magnitude of the vector $n/\lambda$. Set equal to zero, expression (7) represents a $(p+1)$-circular $2(p+1)$-ic curve with singular foci at the roots of the equation

$$\lambda + \sum_{j=0}^{p} \frac{m_j}{z - \alpha_j} = 0.$$

In particular, assuming the hypotheses of Corollary 1, and setting $\sigma_0 - 1 = p = p_0 - 1 = 0$, we find this curve to reduce to the circle $\Gamma_0(z) = 0$. In other words, *if all zeros of a polynomial $f(z)$ of degree $n$ lie in a given circle, any zero of the linear combination $f'(z) + \lambda f(z)$ will lie either in the given circle or in the one obtained by translating the given circle in the direction and magnitude of the vector $n\lambda^{-1}$.**

Finally, by requiring $w = g(z)$, an arbitrary function of $z$, instead of $w = 0$, to satisfy inequality (4), we obtain a theorem similar to Theorem 2 for the zeros of the function $f'(z) + g(z)f(z)$. For example, if $g(z) = \bar{z}$, and if all the zeros of a polynomial $f(z)$ of degree $n$ lie in the circle $|z| \leqq r \leqq 2n^{1/2}$, all zeros of the function $\bar{z}f(z) + f'(z)$ lie in the same circle.

UNIVERSITY OF WISCONSIN,
MILWAUKEE EXTENSION CENTER

* See M. Fujiwara, Tôhoku Mathematical Journal, vol. 9 (1916), pp. 102–108; T. Takagi, Proceedings of the Physico-Mathematical Society of Japan, vol. 3 (1921), pp. 175–179; J. L. Walsh, this Bulletin, vol. 30 (1924), p. 52.