ON CONTINUED FRACTIONS OF THE FORM

$$1 + \overset{\infty}{\underset{1}{K}} (b_\nu z/1)$$

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1. *Introduction.* The principal object of this paper is to determine the region of convergence of the infinite continued fraction

$$1 + \overset{\infty}{\underset{1}{K}} (b_\nu z/1) = 1 + \frac{b_1 z}{1} + \frac{b_2 z}{1} + \cdots, \qquad (b_n \neq 0), \tag{1}$$

when $b_1, b_2, b_3, \cdots$ are real or complex numbers such that for some $k \geqq 1$

$$\lim_{n=\infty} b_{nk+m} = \sigma_m, \qquad (m = 1, 2, 3, \cdots, k). \tag{2}$$

The results may be stated in terms of the numerators and denominators $u_{n,\lambda}$, $v_{n,\lambda}$ of the nth convergent of the continued fraction $1+K_{\nu=1}^{\infty}(\sigma_{\nu+\lambda}z/1)$, $(\sigma_{nk+m}=\sigma_m)$, as follows.

THEOREM 1. *Let*† *us write* $G_k = v_{k-1,1}v_{k-1,2} \cdots v_{k-1,k}$ *and* $H_k = v_k + u_{k-1} - v_{k-1}$; *and let us set*

$$Z(z) = -(-z)^k \sigma_1 \sigma_2 \cdots \sigma_k / H_k^2 .$$

Let R be an arbitrary bounded closed and connected region of the z plane containing the origin on the interior and which contains (within or upon the boundary) none of the zeros of the polynomials G_k, H_k, nor points z such that $Z(z)$ is a real number $\leqq -1/4$. Then (1) converges over R except at certain isolated points $p_1, p_2, \cdots, p_\mu$, and uniformly over the region obtained from R by removing the interiors of small circles with centers $p_1, p_2, \cdots, p_\mu$. The limit is a non-rational function of z analytic over R except at $p_1, p_2, \cdots, p_\mu$, which are poles.

The function $Z(z)$ determines a transformation of the z plane into the Z plane and $Z=Z(z)$. Except in the case $\sigma_1\sigma_2 \cdots \sigma_k = 0$, the set of points in the z plane such that Z is real and

† We write $u_{n,0}=u_n$, and $v_{n,0}=v_n$.

$\leqq -1/4$ is a portion of a curve C_k to which corresponds under the transformation the real Z segment $(-\infty, -1/4)$. The curve C_k is a stelloid† or Holzmüller‡ hyperbola in the plane of $1/z$. In particular, C_1 is a straight line, and C_2 a circle in the plane of z.

In §2 we prove Theorem 1; §3 contains a discussion of the curves C_k; and §4 contains examples and a discussion of the power series which corresponds to (1).

2. *Proof of Theorem* 1. If $N_n(z)/D_n(z)$ is the nth convergent of (1), there are k continued fractions

$$K_m = Z_0^m + \mathop{K}_1^\infty (Z_\nu^m/1), \qquad (m = 1, 2, \cdots, k),$$

with convergents $N_{\nu k+m-1}/D_{\nu k+m-1}$, $(\nu = 0, 1, 2, 3, \cdots)$, and §

$$\begin{aligned} Z_0^m &= \frac{N_{m-1,0}}{D_{m-1,0}}, \\ (3) \quad Z_1^m &= \frac{(-1)^{m-1} b_1 b_2 \cdots b_m z^m D_{k-1,m}}{D_{m-1,0} D_{k+m-1,0}}, \\ Z_n^m &= \frac{-(-z)^k b_1^* b_2^* \cdots b_k^* D_{k-1,(n-3)k+m} D_{k-1,(n-1)k+m}}{D_{2k-1,(n-3)k+m} D_{2k-1,(n-2)k+m}}, \end{aligned}$$

for $n \geqq 3$, and where $b_j^* = b_{(n-2)k+m+j}$ and $N_{n,\lambda}/D_{n,\lambda}$ is the nth convergent of

$$(4) \qquad 1 + \frac{b_{1+\lambda} z}{1} + \frac{b_{2+\lambda} z}{1} + \cdots.$$

By a known theorem,|| if

$$(5) \qquad \lim_{n=\infty} Z_n^m = Z(z)$$

uniformly over a region R', where $Z(z)$ is a continuous function having nowhere in R' a real value $\leqq -1/4$, then there exists an index N such that if $n \geqq N$, the continued fraction $K_{\nu=1}^\infty (Z_{\nu+n}^m/1)$ converges uniformly over R' to an analytic limit $F_n(z)$. Then if

† Fouret, Comptes Rendus, vol. 106 (1888).

‡ Holzmüller, *Einführung in die Theorie der isogonalen Verwandschaften*, 1882, p. 170 and p. 203.

§ Perron, *Die Lehre von den Kettenbruchen*, 1913, p. 200.

|| Ibid., p. 285.

N_n'/D_n' is the nth convergent of K_m, the latter will converge over R' to the limit

$$\frac{N_n' + F_n N_{n-1}'}{D_n' + F_n D_{n-1}'}, \tag{6}$$

provided the denominator in (6) is not $\equiv 0$. But if, as we now suppose, R' contains the origin on the interior, this is impossible because the denominator $=1$ when $z=0$. Hence K_m converges to a function which is analytic except for poles, and clearly converges uniformly in the region obtained from R' by removing the interiors of small circles having these poles as centers. Also, K_m converges uniformly in the vicinity of the origin. From this it follows† that if R' is connected, and if (2) holds for $m=1, 2, 3, \cdots, k$ uniformly over R', and $Z(z)$ is independent of m, then $K_1 \equiv K_2 \equiv \cdots \equiv K_k \equiv P(z)$, where $P(z)$ is the power series corresponding to (1), and hence (1) converges after the manner of K_m to the same value.

It remains to be shown that under the hypothesis (2), (5) holds over the region R described in the theorem, and that $Z(z) = -(-z)^k \sigma_1 \sigma_2 \cdots \sigma_k / H_k^2$. We have, if $\delta_n = nk+m$,

$$D_{2k-1,\delta_n} = D_{k-1,\delta_{n+k}} D_{k,\delta_n} + (N_{k-1,\delta_{n+k}} - D_{k-1,\delta_{n+k}}) D_{k-1,\delta_n}, \tag{7}$$

and hence, if $D_{k-1,\delta_{n+1}}/D_{k-1,\delta_n} = 1+\epsilon_n = 1/(1+\epsilon_n')$,

$$Z_n^m = \frac{-(-z)^k b_1^* b_2^* \cdots b_k^*}{\Delta_{n-3}\ \Delta_{n-2}'},$$

where

$$\Delta_{n-3} = D_{k,\delta_{n-3}} + N_{k-1,\delta_{n-2}} - D_{k-1,\delta_{n-2}} + \epsilon_{n-3} D_{k,\delta_{n-3}},$$

$$\Delta_{n-2}' = D_{k,\delta_{n-2}} + N_{k-1,\delta_{n-1}} - D_{k-1,\delta_{n-1}} + \epsilon_{n-2}'(N_{k-1,\delta_{n-1}} - D_{k-1,\delta_{n-1}}).$$

By (2), $\lim_{n=\infty} \epsilon_n = \lim_{n=\infty} \epsilon_n' = 0$, $\lim_{n=\infty} N_{k,\delta_n} = u_{k,m}$, $\lim_{n=\infty} D_{k,\delta_n} = v_{k,m}$ uniformly over R. Also

$$v_{k,m} + u_{k-1,m} - v_{k-1,m} \equiv v_k + u_{k-1} - v_{k-1} \equiv H_k,$$

for all $m \geqq 1$. It now follows that over the region R

$$\lim_{n=\infty} Z_n^m = -(-z)^k \sigma_1 \sigma_2 \cdots \sigma_k / H_m^2, \qquad (m = 1, 2, \cdots, k),$$

uniformly, as was to be proved.

† Ibid., p. 342. The argument used there applies with slight modification to K_m.

In case $\sigma_1\sigma_2 \cdots \sigma_k=0$, it is clear that $Z(z)\equiv 0$, and therefore can never be real and $\leqq -1/4$. In this case (5) holds uniformly over every bounded region from which the neighborhoods of the zeros of G_k, H_k have been excluded. It may happen that these neighborhoods need not be excluded. More generally, even if (2) fails to hold, we have the following theorem.

THEOREM 2. *If for some integer $k\geqq 1$ the functions Z_n^m, $(m=1, 2, \cdots, k)$, defined in* (3) *converge uniformly to* 0 *for $n=\infty$ over every bounded region, then the continued fraction* (1) *represents a meromorphic function of z and converges except at the poles of that function.*

When $k=1$ this reduces to the condition† $\lim b_n=0$ found by E. B. Van Vleck. In §4 we shall give examples illustrating this theorem in the cases $k=3$, 4.

3. *Discussion of the Curves C_k.* Put $p=(-1)^k\sigma_1\sigma_2\cdots\sigma_k$, and let $p\neq 0$. By C_k' we shall understand the set of points in the z plane which is the image of the real Z segment $(-1/4, -\infty)$ under the transformation

$$Z = -pz^k/H_k^2 .$$

Then C_k' is a portion of a curve C_k which is the image of the negative half of the real Z axis under this transformation, and is a cut for the function represented by (1).

If (2) holds, then

$$\lim_{n=\infty} b_{nq+m} = \sigma_m' , \qquad (m = 1, 2, 3, \cdots, q), \tag{8}$$

where $q=2k$ and $\sigma_{m+k}'=\sigma_m'=\sigma_m$. If we had started with the hypothesis (8), then instead of the function $Z(z)$ we would have $Z'(z)=-p^2z^q/H_q^2$; and

$$G_q = G_k^2H_k^q, \tag{9}$$

$$H_q = H_k^2 - 2pz^k, \tag{10}$$

$$Z'(z) = -\frac{1}{(2+1/Z(z))^2}. \tag{11}$$

In fact, if we let $n=\infty$ in (7), we obtain the relation $v_{q-1,m}=v_{k-1,m}H_k$, from which (9) follows at once. From three relations analogous to (7) we obtain the identities

† Ibid., p. 345.

$$v_q = v_k^2 + v_{k-1}(u_k - v_k), \quad v_{q-1} = v_k v_{k-1} + v_{k-1}(u_{k-1} - v_{k-1}),$$

$$u_{q-1} = u_k v_{k-1} + u_{k-1}(u_{k-1} - v_{k-1}),$$

and consequently $H_q = H_k^2 + 2(u_k v_{k-1} - u_{k-1} v_k) = H_k^2 - 2pz^k$, which is (10). Finally, on eliminating $-pz^k$ between the relations

$$Z = -pz^k/H_k^2, \qquad Z' = -\left[-\frac{pz^k}{(H_k^2 - 2pz^k)}\right]^2,$$

we obtain (11).

It follows from (11) that C_k' is the same as C_q'; and from (9) we see that the zeros of G_q are those of $G_k H_k$. When $p \neq 0$, the zeros of H_k and of H_q lie on the cut; and when $p = 0$, the zeros of H_q are the same as those of H_k by (10). Hence when k is odd in (2) we may turn to (8) instead and obtain precisely the same region of convergence of the continued fraction (1). *There is no loss in generality in assuming k even in* (2).

In order to identify the curves C_k and determine C_k' it will be convenient to replace z by $1/z'$ and study the corresponding curve E_k in the plane of $z' = 1/z$, and the portion E_k' of E_k which corresponds to C_k'. If, as we now suppose, k is even, say $= 2q$, then H_k is a polynomial of degree q of the form $1 + \sum_1^q A_\nu z^\nu$. We find that

$$\frac{(-p)^{1/2}}{[Z(z)]^{1/2}} = z'^q + A_1 z'^{q-1} + \cdots + A_q.$$

As $Z(z)$ ranges through real values $\leqq 0$, $1/(Z(z))^{1/2}$ ranges through pure imaginary values, z' over E_k, and z over C_k. As $Z(z)$ ranges through real values from $-1/4$ to $-\infty$, $1/(Z(z))^{1/2}$ ranges through pure imaginary values from $-2i$ to $+2i$, z' over E_k', and z over C_k'. Set

$$p^{1/2} = \frac{G}{2} e^{i\phi}, \qquad A_\nu = a_\nu e^{i\alpha_\nu}, \qquad z' = re^{i\theta},$$

$$z'^q + A_1 z'^{q-1} + \cdots + A_q = X + iY,$$

where ϕ is any one of the possible arguments of $p^{1/2}$; G, a_ν are real and positive, and α_ν, θ, r, X, Y are real. We have

$$X = \sum_{\nu=0}^{q} a_\nu r^{q-\nu} \cos\,(\alpha_\nu + \overline{q-\nu}\theta),$$

$$Y = \sum_{\nu=0}^{q} a_\nu r^{q-\nu} \sin\,(\alpha_\nu + \overline{q-\nu}\theta),$$

$(a_0=1,\ \alpha_0=0)$. Let t be real. Then E_k is given parametrically by the equations

$$X = t\cos\phi, \qquad Y = t\sin\phi.$$

On eliminating t we find that E_k is the stelloid or Holzmüller hyperbola

$$X\sin\phi - Y\cos\phi = 0, \tag{12}$$

and that E_k' is that part of E_k for which

$$X^2 + Y^2 \leqq G^2. \tag{13}$$

If $q=1$ $(k=2)$, E_2 is a straight line, and (13) is the interior of a circle with center on E_2. The curve C_2' is an arc of a circle, and E_4 is a rectangular hyperbola. In case $q=2$, C_4 and C_4' can be determined by the following special method. First determine δ_1, δ_2 by the conditions

$$\begin{aligned} A_1 &\equiv \sigma_1+\sigma_2+\sigma_3+\sigma_4 = 2(\delta_1+\delta_2),\\ A_2 &\equiv \sigma_1\sigma_3+\sigma_2\sigma_4 = \delta_1^2+\delta_2^2, \end{aligned} \tag{14}$$

so that

$$\delta_\nu = \frac{A_1 + (-1)^\nu(8A_2 - A_1^2)^{1/2}}{4}, \quad (\nu=1,2), \quad 8\,\delta_1\delta_2 = A_1^2 - 4A_2.$$

If $\delta_1\delta_2=0$, then $A_1^2-4A_2=0$. Hence if either A_1 or A_2 is zero, the other is also. If $\delta_1\delta_2\neq 0$, put

$$\Delta = \frac{\sigma_1\sigma_2\sigma_3\sigma_4}{(\delta_1\delta_2)^2},$$

$$Z_1 = -\frac{(\delta_1\delta_2)^2 z^4}{[1+2(\delta_1+\delta_2)z+(\delta_1^2+\delta_2^2)z^2]^2}.$$

Then $Z=\Delta Z_1$. The function Z_1 is of the form of Z' in (11) (for the case $k=4$), so that

$$Z_1 = -1/(2+1/Z_2)^2,$$

where $Z_2 = -\delta_1\delta_2 z^2/(1+\delta_1 z+\delta_2 z)^2$. When $\delta_1\delta_2 = A_1 = A_2 = 0$, it is easy to see that $Z = -\sigma_1\sigma_2\sigma_3\sigma_4 z^4$; and when $\delta_1\delta_2 = 0$, $A_1A_2 \neq 0$, $Z = -\sigma_1\sigma_2\sigma_3\sigma_4 z^4/A_2^2(z+2/A_1)^4$. We find that there are four cases.

CASE 1. $\delta_1\delta_2 = A_1 = A_2 = 0$. The curve C_4' consists of four rays running from z_n to ∞ in the direction from 0 to z_n, where z_n, $(n=1, 2, 3, 4)$, are the four fourth roots of $1/(4\sigma_1\sigma_2\sigma_3\sigma_4)$. In this case $\sigma_1 \neq \sigma_3$ or else $\sigma_2 \neq \sigma_4$, so that the case $k=2$ is not included.

CASE 2. $\delta_1\delta_2 = 0$, $A_1A_2 \neq 0$. In the plane of $z/(z+2/A_1)$ the cut consists of four rays as in Case 1 except that the four fourth roots of $A_2^2/(4\sigma_1\sigma_2\sigma_3\sigma_4)$ are the initial points of the rays. In the plane of z the cut consists of arcs of two circles. Here $\sigma_1 \neq \sigma_3$ or $\sigma_2 \neq \sigma_4$.

CASE 3. $\delta_1\delta_2 \neq 0$, $\delta_1+\delta_2 = A_1/2 \equiv 0$. In the plane of z^2 the cut is an arc of a circle. We may have $\sigma_1 = \sigma_3$, $\sigma_2 = \sigma_4$ if and only if $\sigma_1 = -\sigma_2$. Thus (2) may hold with $k=2$. The cut consists of two rays running to ∞ in this degenerate case.

CASE 4. $\delta_1\delta_2 \neq 0$, $\delta_1+\delta_2 \neq 0$. We may set $\delta_1+\delta_2 = 2\delta$, and apply (11) (for the case $k=2$) to the function $Z_2\delta^2/(\delta_1\delta_2)$. We find that in the plane of $(2+1/(\delta z))^2$ the cut is an arc of a circle. If $\sigma_1 = \sigma_3$, $\sigma_2 = \sigma_4$, $\sigma_1 \neq -\sigma_2$, (2) holds with $k=2$, and the cut is an arc of a circle in the plane of z. If $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4$, (2) holds with $k=1$ and the cut is the ray running from $-1/(4\sigma_1)$ to ∞ in the direction from 0 to $-1/(4\sigma_1)$.

If the σ_n are real and positive it is easy to show that $\delta_1\delta_2 \neq 0$, $\delta_1+\delta_2 \neq 0$, so that Case 4 obtains. In this important case one may show that the cut is a portion of the negative half of the real z axis. The polynomial G_4 is

$$(1 + \sigma_1 z + \sigma_2 z)(1 + \sigma_2 z + \sigma_3 z)(1 + \sigma_3 z + \sigma_4 z)(1 + \sigma_4 z + \sigma_1 z).$$

4. *Examples and Applications.* The following examples have been selected for the purpose of bringing out interesting points which might otherwise be overlooked.

EXAMPLE 1. Let $b_{3n-2} = c_n \neq 0$, $\lim c_n = 0$, $b_{3n+2} = -b_{3n} = a \neq 0$. Here we have, with $k=3$,

$$Z_n^1 = -\frac{z^3a^2c_n}{(1 + c_{n-1}z)(1 + c_n z)},$$

$$Z_n^2 = -\frac{z^3a^2c_n(1 + c_{n-1}z)(1 + c_{n+1}z)}{[1 + (c_n + c_{n-1})z][1 + (c_n + c_{n+1})z]},$$

$$Z_n^3 = -\frac{z^3a^2c_n}{(1+c_nz)(1+c_{n+1}z)}.$$

Since $\lim_{n=\infty}Z_n^m=0$, $(m=1, 2, 3)$, uniformly over every bounded region, the continued fraction represents a meromorphic function of z by Theorem 2.

EXAMPLE 2. Let $b_{4n+1}=-b_{4n+2}=c_n>0$, $\lim c_n=0$, $b_{4n}=-b_{4n-1}=a>0$. If $b_1=1/a_1$, $b_n=1/(a_{n-1}a_n)$, $(n\geqq 2)$, then $a_{2n+1}>0$. Since $\sum a_{2n+1}$ diverges, it follows from the work of Hamburger† that the continued fraction converges except upon the real axis. To prove that it represents a meromorphic function‡ it is but necessary to note§ that when $k=4$

$$Z_n^1 = -\frac{z^4a^2c_{n-1}c_{n-2}}{(1-ac_{n-2}z^2)(1-ac_{n-1}z^2)},$$

so that $\lim Z_n^1=0$ uniformly over every bounded region, and therefore K_1 represents a meromorphic function.

EXAMPLE 3. If $\lim\sup|b_n|<g$, $\lim_{n=\infty}b_nb_{n+1}=0$, we may show that (1) represents a function which is analytic except for poles in the region‖ $|z|\leqq 1/(2g)$. In fact, if $k=2$, $\lim_{n=\infty}Z_n^m=0$, $(m=1, 2)$, uniformly over this region inasmuch as the polynomials $D_{3,2n+m}=1+(b_{2n+m+2}+b_{2n+m+3})z$ have no zeros in this region if n is sufficiently large; and $\lim_{n=\infty}b_1^*\,b_2^*=0$.

EXAMPLE 4. According to Theorem 1, the real segment $(1, \infty)$ is to be excluded from the region of convergence of the continued fraction (1) if the b's have the values $b_1=1/2$, $b_2=-1/2$, $b_{2n}=-1/(2(1+[n-1]^p/n^p))$, $b_{2n+1}=-1/(2(1+[n+1]^p/n^p))$. One may show that this continued fraction converges or diverges at $z=1$ (a point on this segment) according as p is >1 or $\leqq 1$. In fact, when $z=1$, the nth convergent is

† Mathematische Annalen, vol. 82, pp. 120–187.

‡ See this Bulletin, vol. 39 (1933), pp. 946–952, in which another example is given to show that (1) may represent a meromorphic function when the b_n are real and $b_{2n}b_{2n+1}>0$, $\lim\sup|b_n|>0$. In that example convergence was established except at the poles of the function, whereas here the question of convergence at points on the real axis is not considered.

§ The other Z_n^m, $(m=2, 3, 4)$, are not all so simple in character.

‖ If the condition $\lim b_nb_{n+1}=0$ is dropped, the same holds in the region $|z|\leqq 1/(4g)$ (see Perron, loc. cit., p. 343).

$$\frac{1}{1^p}+\frac{1}{2^p}+\cdots+\frac{1}{[n/2]^p}$$

when n is even and

$$\frac{1}{1^p}+\frac{1}{2^p}+\cdots+\frac{1}{[(n-1)/2]^p}+\frac{1}{2[(n-1)/2]^p}$$

when n is odd.

As is well known, (1) has a unique corresponding power series $P(z)=\sum c_\nu z^\nu$, $(c_0=1)$, from which (1) may be obtained by a repeated division process. If a given power series $P(z)$ has a corresponding continued fraction of the form (1), it is said to be *semi-normal.*† When convergent, the continued fraction furnishes a method for summing the power series. Let $P'(z)=\sum_{\nu=0}^{\infty}c_{1+\nu}Z^\nu$ be semi-normal with corresponding continued fraction $c_1+K(b_\nu' z/1)$. Then if (1) converges to $f(z)$ when (1) corresponds to $P(z)$, it is important to know conditions under which $c_1+K(b_\nu' z/1)$ converges to the value $(f(z)-1)/z$.

It is well known that if (1) converges to $f(z)$, then when $c_1+K(b_\nu' z/1)$ converges it must have the value $(f(z)-1)/z$. This follows from the fact that the even convergents of $1+c_1z+zK(b_\nu' z/1)$ are the same as the odd convergents of (1).‡

The numbers b_ν and b_ν' are related as follows.§ Set $a_1=1/b_1$, $a_1'=1/b_1'$, $a_n=1/(b_na_{n-1})$, $a_n'=1/(b_n'a_{n-1}')$, $(n>1)$. Then if $h_n=a_1+a_3+\cdots+a_{2n+1}$,

$$a_{2n}'=a_{2n+1}/(h_nh_{n-1}),\qquad a_{2n+1}'=a_{2n+2}h_n^2\,.$$

If (2) holds with k even (say $=2q$), then it is not difficult to show that when the b_n are real and

$$(15)\qquad \lim_{n=\infty}(b_{nk+m}/b_{nk+m+1})=r_m>0,\qquad (m=1,2,3,\cdots,k),$$

we must also have

$$(16)\qquad \lim_{n=\infty}(b_{nk+m}'/b_{nk+m+1}')=r_m'>0,\qquad (m=1,2,3,\cdots,k),$$

† Perron, loc. cit., p. 304.

‡ Perron, loc. cit., p. 447.

§ Transactions of this Society, vol. 31 (1929), pp. 102–103.

and

$$\lim_{n=\infty} b'_{nk+m} = \sigma'_m, \qquad (m = 1, 2, 3, \cdots, k). \tag{17}$$

Hence if Theorem 1 is applicable to the continued fraction corresponding to $P(z)$ it is also applicable to the continued fraction corresponding to $P'(z)$ provided $P(z)$ has real coefficients and (15) holds. From (16), (17) it then follows at once that Theorem 1 can be in turn applied to the continued fraction corresponding to $P''(z)=\sum_{\nu=0}^{\infty} c_{2+\nu}z^\nu$, provided the latter is semi-normal, etc. It is easy to conclude from the fact that two successive continued fractions obtained in this way have an infinite number of convergents in common that the function $Z(z)$ of Theorem 1 is the same for all these continued fractions. We shall summarize the result in the following theorem.

THEOREM 3. *If $P(z)=1+c_1z+c_2z^2+\cdots$ is a semi-normal power series with real coefficients and corresponding continued fraction* (1) *such that for $k=2q$ equations* (2) *and* (15) *hold, and if $L=c_1+K(b'_\nu z/1)$ is the corresponding continued fraction for $P'(z)=c_1+c_2z+c_3z^2+\cdots$ (supposed semi-normal), then* (1) *and $1+zL$ converge to one and the same function $f(z)$ over the region R described in Theorem* 1. *If $P^{(n)}(z)=\sum_{\nu=0}^{\infty} c_{n+\nu}z^\nu$ is semi-normal with corresponding continued fraction $c_n+K(b_\nu^{(n)}z/1)$, $(n=1, 2, 3, \cdots, r)$, then all the continued fractions $1+c_1z+\cdots+c_nz^n+z^nK(b_\nu^{(n)}z/1)$, $(n=1, 2, 3, \cdots, r)$, converge over R to $f(z)$.*

The continued fractions are precisely the continued fractions of "type 1" of a Padé table,† whose convergents are "stairlike" files of approximants beginning upon the horizontal side of the table. One can show that when $E(z)$, the reciprocal of $P(z)$, and the series $E^{(n)}(z)$ obtained by removing from $E(z)$ the first n terms and the factor z^n, $(n=1, 2, 3, \cdots)$, are semi-normal, then under the hypothesis of Theorem 3 the continued fractions corresponding to stairlike files of approximants beginning on the vertical side of the table also converge over R to $f(z)$.

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† Perron, loc. cit., pp. 447–448.