ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

292. Dr. F. G. Dressel: *A note on a linear operational equation.*

The paper presents solutions of the equation $L(u)+f(x)P(u)=0$, where $L(u)$ is a functional operator with properties similar to those of a homogeneous linear differential operator, and $P(u)$ is a linear operator. When the methods of this paper are applied to the problem considered by R. P. Gillespie (Proceedings of the Edinburgh Mathematical Society, (2), vol. 4 (1935), pp. 80–84) more complete results are obtained than those presented in Gillespie's paper. (Received July 24, 1935.)

293. Dr. J. S. Frame: *The simple group of order* 25920. Preliminary report.

This paper extends the results of a paper *On the irreducible representations of hyperorthogonal groups* (presented to the Society December 28, 1934 and soon to appear in the Duke Mathematical Journal). A certain infinite family of simple groups, $HO(m, q^2)$, may be represented by unitary matrices of degree m, with coefficients from a finite field $GF(q^2)$ of q^2 elements, where q is the power p^s of a prime p, and conjugates are defined by $\bar{x}=x^q$. The vectors a with components a_i from the $GF(q^2)$, for which $\sum a_i a_i=0$, are permuted among themselves and so are those for which $\sum a_i a_i=k\neq 0$. A set of monomial representations, which were found to be irreducible when $m=3$ are no longer so when $m>3$, but always split into 2 or 3 irreducible components. Only one of the simple groups in question is of order less than three million. This is the group $HO(4, 4)$, of order 25920. It can be generated by two operators of orders 2 and 5 respectively, whose product is of order 9. It is analyzed into its 20 sets of conjugate operations, the characters of 6 reducible monomial representations are split into irreducible components, and finally the complete table of characters for the group is obtained. (Received July 29, 1935.)

294. Mr. C. B. Tompkins: *An integral invariant of n-dimensional shells in* $(2n-1)$*-space.*

A method is presented for determining an integral invariant associated with a homeomorph of an n-dimensional sphere lying in euclidean $(2n-1)$-

space. An integer is obtained by means of an integration over a contour, an $(n-1)$-dimensional simple closed variety which does not bound on the shell. For a special set of standard shells the invariant is shown to represent the number of twists of the shell, the number of times any pair of non-intersecting contours interlink. The particular problem is to start with the shell defined locally by means of a generalization of the Gauss formulas, and to obtain from the coefficients of these formulas an integral invariant which is determined by the shell. The invariant was found by Rainich when $n=2$. The method depends upon finding the solid angle described by the vector lying in the shell perpendicular to a contour, this vector referred to invariant directions of the contour, while its base point passes over the contour. The invariant directions are the mean curvature vector and vectors obtained by repeated application of the generalized Laplacian, depending on coefficients of the Gauss formulas as connection coefficients, on this mean curvature vector. (Received August 1, 1935.)

295. Dr. C. W. Vickery: *Concerning distance functions and point functions.*

In the present treatment is considered a generalization of the notion of distance satisfied by various connected spaces, including certain spaces in which there exist uncountable convergent sequences of points. Suppose that with each pair of points (A, B) is associated a type μ sequence $\delta_\mu(AB)$ of positive real numbers (i.e., a point of space E_μ) satisfying certain conditions analogous to metric conditions of Fréchet; such a space is called a space Δ_μ'. It is shown that spaces E_μ, Δ_μ, and $U_{\eta_1\omega_1}$ are spaces Δ_μ', Δ_μ', and Δ_{ω_1}' respectively. Necessary and sufficient conditions for spaces Δ_μ' in terms of point and region are considered. Also functions are considered which map a point set P on a point set Q in spaces Δ_{ω_α}'. Oscillation is defined such that a function is continuous at p_0 if and only if its oscillation at p_0 is ϕ $(0, 0, 0, \cdots)$. The set of points of discontinuity is a set $F_\sigma\aleph_\alpha$ relative to P (sum of $\aleph_\alpha$ sets closed relative to P) and the set of points of continuity is a $G_\delta\aleph_\alpha$ relative to P (product of $\aleph_\alpha$ open subsets of P). (Received August 2, 1935.)

296. Dr. C. W. Vickery: *Spaces of uncountably many dimensions.*

In this treatment are considered spaces D_ν, E_ν, and E_ν', the set of all points of which is the set of all type ν sequences of real numbers, ν being an ordinal number. The definitions of sequential limit point are the same for spaces D_ν and E_ν as for spaces D_ω and E_ω, respectively, of Fréchet. It is shown that for every ν, D_ν and E_ν are arc-wise and locally arc-wise connected. If ν is an element of $Z(\aleph_\alpha)$, then D_ν is homeomorphic with D_{ω_α} and E_ν is homeomorphic with E_{ω_α}. For every ν, D_ν is metric. Space D_{ω_α} is not $\aleph_x$-separable for any $\aleph_x<2^{\aleph_\alpha}$. For $\nu\geqq\omega_1$, E_ν is not metric. Region is defined in space E_{ω_α}, and limit point in terms of region, in such a way that the original meaning of sequential limit point is preserved; this extends the notion of limit point so that there are uncountable convergent sequences of points (for $\nu\geqq\omega_1$); the resulting space is called E_{ω_α}'. Space E_{ω_α}' is a distributive space $\Gamma(\aleph_\alpha)$ (see author's paper, Tôhoku Mathematical, Journal vol. 40 (I) (1935), pp. 1–26). (Received August 2, 1935.)

297. Professor A. T. Craig: *Note on a certain bilinear form that occurs in statistics.*

In this paper, we investigate some of the properties of the distribution of a real symmetric bilinear form of $2n$ variables normally and independently distributed. (Received August 4, 1935.)

298. Professor C. C. Craig: *A new exposition and chart for the Pearson system of frequency curves.*

By making the quantities α_3 and δ (*The Handbook of Mathematical Statistics*, H. L. Rietz, Editor-in-Chief, p. 104) fundamental in the discussion, a new exposition of the Pearson system of frequency functions is developed which the author believes possesses marked advantages in unity, clarity, elegance, and convenience. The chart corresponding to the Rhind-Pearson diagram is strikingly simple and clear. (Received August 5, 1935.)

299. Professor C. C. Craig: *Sheppard's corrections for a discrete variate.*

By the use of moment generating functions and the argument used by R. A. Fisher in the case of a continuous variable, the corrections to the moments derived by J. R. Abernethy (Annals of Mathematical Statistics, vol. 4 (1933), pp. 263–277) for grouping in the case of a single discrete variable are derived in a considerably more elegant manner. The method is immediately adapted to two and more variables and the results likely to be useful are given for two variables. The method gives first the corrections to the semi-invariants (of Thiele) which are much simpler in form. The known corrections for continuous variables are obvious limiting forms of the results given. (Received August 5, 1935.)

300. Dr. Ben Dushnik: *Some theorems on the maximum sum of ordinals.*

Consider a sum of the type $\Sigma \equiv \Sigma_{\beta<\alpha} a_\beta$, where α is a specified ordinal number and, for every ordinal $\beta<\alpha$, a_β is likewise a specified ordinal number. The ordinal α is called the argument of the sum Σ, while the set of different ordinals which appear as summands in Σ, each with its "multiplicity," may be called the "system of summands" of the sum Σ. Two systems of summands are identical only if they comprise the same set $\mathfrak{A}$ of ordinals, and if each ordinal of $\mathfrak{A}$ has the same multiplicity in both systems. When one specifies the argument α and the system of summands, one may still obtain different sums Σ corresponding to different arrangements. Thus, the two sums $\Sigma_1 = \omega^4+\omega^3+\omega+\omega+\cdots+\omega$ and $\Sigma_2 = \omega^3+\omega+\omega+\cdots+\omega^4$, where ω is the least transfinite ordinal, have the same argument, $\omega+1$, and the same system of summands: ω^4, ω^3, ω, with multiplicities 1, 1, $\aleph_0$, respectively; yet $\omega^4+\omega^3+\omega^2+\omega = \Sigma_1 > \Sigma_2 = \omega^4$. The present note considers whether, for a given argument and system of summands, there exists a "maximum" sum Σ, and if so, what arrangement of summands gives this maximum (in the above example, Σ_1 is the greatest sum). (Received August 5, 1935.)

301. Professor M. H. Ingraham: *Characteristic spaces associated with the matrix whose elements belong to a division algebra.*

If M is a matrix with coefficients in a field F, then the rank of M and of $N=T^{-1}MT$ are equal, and for every polynomial g the rank of $g(M)$ is equal to the rank of $g(N)$ and the space of vectors ξ such that $g(M)\xi=0$ has the same rank as the space of vectors η such that $g(N)\eta=0$. If, however, M, T, N, and ξ have elements belonging to a division algebra or to a quasi-field D, and if $g(x)=\sum x^i d_i$ where the d's are in D, then the above theorem no longer holds. A process of multiplication of such polynomials and an operation with such polynomials, and associated matrices upon vectors is defined so as to reestablish much of the theory connected with matrices and their characteristic values, characteristic vectors and invariant factors and associated spaces. This theory leads to great simplifications of the theory for the similarity of matrices whose elements belong to a division algebra. (Received August 5, 1935.)

302. Dr. W. T. Martin (National Research Fellow): *Geometrical representation of the growth of entire functions of several complex variables.*

We say that the function $F(z_1, z_2, \cdots, z_n)=\sum_{m_1,\cdots,m_n=0}^{\infty} a_{m_1\cdots m_n} z_1^{m_1}\cdots z_n^{m_n}$ is of *exponential type* if the associated series defined by the formula $f(z_1, \cdots, z_n) =\sum_{m_1,\cdots,m_n=0}^{\infty} a_{m_1\cdots m_n}[(m_1+\cdots+m_n)!]z_1^{m_1}\cdots z_n^{m_n}$ possesses a ($2n$-dimensional) region of convergence. We obtain for such functions F analogues of certain of the results obtained by G. Pólya (Mathematische Zeitschrift, vol. 29 (1929), pp. 549–640) for functions of a single variable. With every function F of exponential type we associate a real-valued function $h(\alpha_1, \cdots, \alpha_n)$, ($\alpha_1, \cdots, \alpha_n$ arbitrary complex numbers), which measures the growth of the function F in the direction of the half ray which joins the origin to the point $(\alpha_1, \cdots, \alpha_n)$. By means of the function h we associate with the function F a uniquely determined open region D in the space R_{2n}. The region D contains the invariant convergence region of the power series development of the function f and is an "invariant" region (invariant under linear transformations of coordinates!). Every analytic (2-dimensional) plane P through the origin cuts D in a region in which f is regular and which is the "best" region in P (of a given character) in which f is regular. (Received August 5, 1935.)

303. Professor A. D. Michal: *Notes on Riemannian and non-Riemannian differential geometry in abstract spaces.*

Sets of postulates for Riemannian and non-Riemannian differential geometry in abstract spaces were first given by the author in 1933. The consequences of these postulates were developed sufficiently far to yield a theory of parallel displacement and curvature. In the present paper new postulates are added to some modified sets of postulates and we study such subjects as abstract Riemannian geometry with torsion, abstract Riemannian and non-Riemannian geometries with distant parallelism, and abstract geometries with a *class* of generalized Ricci tensors. It is worth while to mention here that there exist infinitely dimensional functional geometries with a *continuous infinitude* of generalized Ricci functional tensors. (Received August 5, 1935.)

304. Dr. J. F. Randolph: *On generalizations of length and area.*

Definitions of p-dimensional measure of q-dimensional sets have been given by C. Carathéodory, W. Gross, and others. In this paper is discussed, without answering it completely, the simplest phase of the fundamental question whether the generalizations of length and area under these definitions preserve, as do Lebesgue's, the euclidean relation that area is the product of length by length. Let B be a plane point set and A the set of all points (x, y, z) such that (x, y) is in B and z lies between zero and a constant h. If B is C-linearly measurable we show that B is also G-linearly measurable (the converse is not true), that the space set A is both C- and G-plane measurable, and furthermore that the C-plane measure of A is less than or equal to h times the C-linear measure of B and the inequality is reversed for G-measures. Consequently for the restricted class of sets B for which the C- and G-linear measures are the same, the "area" of A is the "base" times the altitude. Finally we obtain a condition for the equality of the C- and G-linear measures of B, but show that this condition is not necessary. (Received August 5, 1935.)

305. Miss V. E. Spencer: *A classification of certain sequences of polynomials associated with persymmetric determinants.*

A sequence of polynomials $\{\Phi_i(x)\}$ is associated with any sequence of persymmetric determinants $\{\Delta_i\}$ by replacing the last row of Δ_i by 1, x, $x^2, \cdots, x^{i-1}$. Let the numerical value of $\{\Delta_i\}$ be $\{\delta_i\}$. A set S is defined as a set of sequences $\{\Phi_i(x)\}$ with each of which is associated the same $\{\delta_i\}$. If a set S contains $\{\Phi_i(x)\}$ it contains also $\{\Phi_i(x-m)\}$, m arbitrary. Any set S contains one and only one symmetric sequence. A study is made of the matrices transforming one sequence of S into another. Invariants of S are found. Orthogonal Tchebycheff polynomials (O.T.P.) form a subclass of all sets S. In connection with the theory new bounds for the zeros of O.T.P. are exhibited. (Received August 5, 1935.)

306. Mr. John W. Wrench, Jr.: *Sequences of transformations in a general metric space.*

This paper is concerned with the general problem: If two sequences of point sets (M_i) and (N_i) having sequential limiting sets M and N respectively are given in a metric space, and if, for each i, T_i is a transformation such that $T_i(M_i)=N_i$, then under what conditions on T_i, M_i, and N_i does there exist a transformation T such that $T(M)=N$? Under sufficiently strong uniformity conditions, the transformation T is a homeomorphism; under weaker conditions T is (1–1) and continuous, but its inverse may not be continuous; under still weaker conditions it can merely be stated that T is (1–1). The paper is an extension of one by H. M. Gehman (Proceedings of the National Academy of Sciences, vol. 18 (1932), pp. 460–465). (Received August 5, 1935.)

307. Professor M. M. Culver: *Fundamental regions in S_{12} for the simple collineation group of order* 1092 *on seven variables.*

A certain set of fourteen Hermitian forms are selected which are permuted by the operators of this group. The ninety-one differences of these forms, equated to zero, are regarded as hypersurfaces in S_{12}. A point which lies on exactly seventy-eight of these hypersurfaces is selected. In a sufficiently small neighborhood of this point there are exactly 13! regions. From this fact it can be shown that the whole set of hypersurfaces divides the S_{12} into 14! regions. These 14! regions may be regarded as consisting of 1092 sets of $2 \cdot 11!$ each. Thus there are 1092 fundamental regions in S_{12} for the group in question. Of seven complex variables there exist sets of fourteen linear forms whose absolute values, for suitably chosen values of the variables, may be given any desired arrangement in order of magnitude. (Received August 6, 1935.)

308. Dr. J. H. Curtiss: *Interpolation in non-regularly distributed points.*

In a paper to appear in the Transactions of this Society (see abstract 40-9-263) the author studied the convergence of the sequence of polynomials which coincide with an integrable function in the sequence of "regularly distributed" sets of points on a Jordan curve C. This present note discusses the use of other possible choices of the points of interpolation in this problem, and in particular, describes in some detail the results obtained when the points are transforms of the roots of unity under maps other than the map which transforms the roots of unity into regularly distributed points on the curve C. (Received August 6, 1935.)

309. Mr. I. E. Highberg, Professor A. D. Michal, and Mr. A. E. Taylor: *Abstract euclidean spaces.* III.

In this paper the authors continue their study of a certain abstract space E with independently postulated "geometric" and "analytical" metrics (see abstracts 40-11-383 and 41-3-158). A set of sixteen independent postulates is given. The co-ordinate methods of Hilbert space are not applicable and several results, trivial in Hilbert space, take on an important and difficult aspect. The following theorem, among others, is proved on rotations: Let $y=x+T(x)$ be a proper rotation, and let $\Gamma(x)$ be the transformation associated with the inverse, $x=y-\Gamma(y)/2$, of $y=x+T(x)/2$; then $\Gamma(x)$ is "skew-symmetric." Conversely, if $\Gamma(x)$ is a "skew-symmetric" transformation, such that $x=y-\Gamma(y)/2$ has the unique solution $y=x+T(x)/2$ for each x in E, then $x+T(x)$ is a proper rotation. Some of the key theorems and methods of Hilbert space as developed by Hilbert, Riesz, Stone, von Neumann, and others in connection with projection operators, characteristic value problems, and resolution of the identity do not go over to abstract euclidean spaces. Several of the theorems that go over to E are proved by new methods. (Received August 6, 1935.)

310. Professor A. D. Michal and Mr. D. H. Hyers: *An existence theorem for a second order differential system with two-point boundary conditions in general analysis.*

In this paper an existence theorem for the differential equations $d^2x(t)/dt^2 = F(t, x, dx/dt)$ in complete normed vector spaces is proved. An application is then made to the theory of geodesics in Riemannian and non-Riemannian geometries in abstract spaces. Some attention is also given to function space instances of the general theory. (Received August 6, 1935.)

311. Professor D. W. Woodard: *Sufficient conditions that a space be an n-dimensional manifold.*

This paper is based upon results obtained by the author in a preceding paper (see abstract 40-7-236). For definitions of the terms and symbols employed in the theorem which follows, the reader is referred to the above-mentioned paper which has been submitted to the Transactions for publication. The present work, which makes use of a result obtained by G. Nöbeling (Monatshefte für Mathematik and Physik, vol. 42, p. 145), is concerned with the proof of the following theorem: A θ-space M^n is an n-dimensional manifold provided the following conditions are fulfilled: (1) There exists at least one neighborhood $N_{\alpha\beta}$ such that, if there is a homeomorphism, $H_1[\lambda(N_{\alpha\beta})]=\lambda(N_{\alpha\beta})$, then there exists a homeomorphism, $H_2(\overline{N}_{\alpha\beta})=\overline{N}_{\alpha\beta}$, of such nature that $H_2[\lambda(N_{\alpha\beta})]=H_1[\lambda(N_{\alpha\beta})]$. (2) Let W_1 and W_2 be two sets such that $W_1\cdot W_2\neq 0$, $W_i\cdot \not\supset W_j$, $i\neq j$, W_1 is a neighborhood $N_{\phi\psi}$, W_2 is an element of the set $\Delta_{i=1}^{a}[P_i]N_{\gamma\delta}$, where $0\subseteq P_i\subseteq\sum_{k=1}^{b}\lambda(N_{\alpha_k\beta_k})$, $\alpha_k\leqq\gamma$, and, if $W_2\neq N_{\gamma\delta}$, $\phi\leqq\gamma$. Then, $\overline{W}_2\cdot\lambda(W_1)=\sum_{j=1}^{t}\overline{C}_j^{n-1}$, $\Delta_{j=1}^{t}[C_{\beta_j}^{n-1}]W_2\neq 0$ and every component of $\lambda(W_2)-\lambda(C_i^{n-1})$ is an $(n-1)$-cell. (Received August 6, 1935.)

312. Professor H. A. Davis and Dr. Amos Black: *Some non-involutorial space transformations associated with pencils of nodal cubic surfaces.*

Transformations defined by means of a one-to-one correspondence between the surfaces of two pencils of nodal cubic surfaces and the points of certain rational curves are discussed. Their association with certain transformations of Pieri and Caldarera is shown. (Received July 31, 1935.)

313. Dr. Amos Black: *A series of involutorial Cremona space transformations defined by a pencil of ruled cubic surfaces.*

A series of transformations is defined by means of a correspondence between the surfaces of a pencil of ruled cubic surfaces and the points of certain rational curves, called the director curves. The director curve is not part of the basis of the pencil; one of the principal surfaces is a ruled surface R, all of whose generators are parasitic lines; all the surfaces of the homoloidal web have fixed tangent planes along a certain curve, but none of the fixed planes is determined by R. (Received July 8, 1935.)

314. Professor Hassler Whitney: *Differentiable manifolds in euclidean space.*

If M is an m-dimensional manifold defined by overlapping neighborhoods, and if the relation between the two given sets of coordinates in the common part of two neighborhoods is one with continuous derivatives through the rth order ($r\geqq 1$) and with non-vanishing Jacobian, we say M is of class C^r.

We show that M may be mapped in a 1–1 differentiable manner (in fact in a "C^r-manner") in euclidean space E^{2m+1}. Any such M in any E^n may be approximated to by an analytic manifold in E^n, the approximation being closer and closer as we approach the boundary (if there is one). Hence M may be given an analytic Riemannian metric. An outline of the paper will be found in the Proceedings of the National Academy of Sciences for July. (Received July 8, 1935.)

315. Professor C. G. Latimer: *Note on the quadratic subfields of a generalized quaternion algebra.*

Let $\mathfrak{A}$ be a rational generalized quaternion algebra with the fundamental number d. Since every element of $\mathfrak{A}$ satisfies a quadratic equation with rational coefficients, $\mathfrak{A}$ contains an infinitude of quadratic fields. In this note it is shown that necessary and sufficient conditions for $\mathfrak{A}$ to contain a given quadratic field F are: (a) F is imaginary if $d>0$, and (b) no rational prime factor of d is the product of two distinct prime ideals in F. Fueter showed that (b) is a necessary condition (*Quaternionenringe*, Commentarii Mathematici Helvetici, vol. 6 (1934), p. 199). (Received July 11, 1935.)

316. Dr. W. C. Randels: *On Volterra-Stieltjes integral equations.*

An existence theorem is proved for the integral equation $f(x)=g(x)+\int_0^x f(y)\, d_yK(x, y)$. The principal restriction on $K(x, y)$ is that there exists a monotone function $V(y)$ such that $|K(x, y_2)-K(x, y_1)| \leqq |V(y_2)-V(y_1)|$. An example is given to show that this restriction cannot be replaced by $\int_0^1 |d_yK(x, y)| <1$. (Received July 11, 1935.)

317. Dr. W. C. Randels: *On a theorem of Plessner.*

Plessner has shown that if $f(x) \subset L_2$ and $\{a_n\}$, $\{b_n\}$ are the Fourier coefficients of $f(x)$, then $\sum_{n=2}^{\infty}(a_n \cos nx+b_n \sin nx)/\sqrt{\log n}$ converges almost everywhere. If we denote by $E(Pl,f)$ the set where Plessner's series converges, then we show that $E(Pl,f) \not\subset E(L,f)$, $E(L,f) \subset E(Pl,f)$ where $E(L,f)$ is the Lebesgue set of $f(x)$. (Received July 11, 1935.)

318. Professor G. Szegö: *Inequalities for the zeros of Legendre polynomials and related functions.*

Denoting the zeros of the Legendre polynomial $P_n(\cos\theta)$ in $(0, \pi)$ (in increasing order) by $\theta_1, \theta_2, \cdots, \theta_n$, Bruns, and later in a sharper form, A. Markoff and Stieltjes (independently) have given various estimates for θ_v. In the first part of the present paper Sturm's classic method is used to obtain the inequalities $\pi(v-1/4)/(n+1/2)<\theta_v<v\pi/(n+1)$, $v\leqq g=[n/2]$. The upper bound is the same as that of Markoff-Stieltjes, the lower one is even better. As a consequence of Sturm's method, the convexity of the sequence $0=\theta_0, \theta_1, \theta_2, \cdots, \theta_{g+1}$ is stated, this fact being used in the proof of the inequalities mentioned. Sturm's method leads also to various other estimates of the zeros of P_n as well as of related functions. The second part of the paper deals with a

general distribution theorem for the zeros of the trigonometric polynomials $\sum_{k=0}^{m} \lambda_k \cos(m-k)t$ with $\lambda_0 > \lambda_1 > \cdots > \lambda_m > 0$. According to Pólya these are all real and simple. Now, each interval $\pi(v-1/2)/(m+1/2)$ $\pi(v+1/2)/(m+1/2)$, $(v=1, 2, \cdots, m)$, contains exactly one of these zeros. This gives an estimate for θ_v also, although not as good an estimate as Sturm's method. It can, however, be improved by direct treatment. (Received July 11, 1935.)

319. Professor Alonzo Church and Dr. S. C. Kleene: *Formal definitions in the theory of ordinal numbers.*

The theory of formal definition (λ-definition) of functions of positive integers, developed by Kleene (American Journal of Mathematics, vol. 57 (1935), pp. 153–173, 219–244), is extended to the transfinite ordinals. Formulas are assigned to represent the ordinals in the first number class and those in a certain subset of the second number class, plausibly identified as the set of "constructive" ordinals in the second number class. To each ordinal in this subset of the second number class, however, an infinite number of non-interconvertible formulas is assigned, and there is no means by which it can be effectively determined, in general, whether two given formulas represent the same ordinal. The notion of formal definition of functions of ordinals is introduced in the manner of Kleene, p. 219, and the formal definability of certain familiar functions of ordinals is established. It is pointed out that, in the case of any formally definable function of ordinals, the process of reduction of a formula to normal form provides an algorithm for calculating the values of the function (in the sense of calculating formulas which represent those values). In conclusion a method is outlined of extending the theory to greater ordinals. (Received July 13, 1935.)

320. Professor T. R. Hollcroft: *Properties of branch-point manifolds associated with linear systems of primals.*

A linear r-parameter system of primals (hypersurfaces) F of order n in S_r is associated with an involution whose properties and those of F are determined by the characteristics of the branch-point primal L of the involution. The following theorem is proved: The characteristics of the tangent cone from S_k to L, the branch-point primal associated with a linear r-parameter system of primals in S_r, are respectively the characteristics of the branch-point manifold L_{r-k-2} associated with a linear $(r-k-1)$-parameter system of primals contained in the linear r-parameter system. The values in terms of n of the order and all $r-1$ classes of L are obtained and the classes of all sections of L by linear manifolds. (Received July 15, 1935.)

321. Professor H. L. Rietz: *On the frequency distributions of certain ratios.*

Certain properties of the frequency distributions of the ratios y_i/x_i, when x_i and y_i are drawn from normal universes, have been the subject of study of several authors, notably Greenwood, Karl Pearson, C. C. Craig, and Greary. The present paper continues the study by finding the frequency functions of $t=y_i/x_i$ where x_i and y_i are taken at random from very simple distributions limited in several cases to a finite part of the xy-plane, and noting some of their properties. (Received July 18, 1935.)

322. Professor Gordon Pall: *Binary quadratic discriminants differing by square factors.*

The Lipschitz correspondence, by means of linear transformations of determinant p, between binary quadratic forms of discriminants d and p^2d is combined with precise relations among the numbers of sets of representations of n and p^2n in a class, genus, or order. Incidentally the number of ambiguous classes derivable from a given ambiguous class is determined. (Received July 19, 1935.)

323. Professor F. Riesz and Dr. E. R. Lorch: *The integral representation of unbounded self-adjoint transformations in Hilbert space.*

The paper deals with unbounded self-adjoint transformations in Hilbert space. It embodies two short proofs of the formula $A = \int_{-\infty}^{\infty} \lambda dE(\lambda)$ for such a transformation. Previous proofs have all been based on the theory of bounded transformations. But none of these makes systematic use of the theory of such transformations, specifically to the extent of making use of the above formula and its immediate consequences for bounded transformations. The point of view of the paper is to derive the formula, exploiting as fully as possible results concerning bounded transformations well known for over twenty years. The first proof demonstrates the existence of $E(\lambda)$, the so-called resolution of the identity of A, but does not allow of a simple construction for it. The second proof gives a construction of $E(\lambda)$ in terms of elementary operations on A. (Received July 23, 1935.)

324. Dr. F. G. Dressel: *Solutions of bounded variation of the Volterra-Stieltjes integral equation.*

The purpose of this paper is to give conditions under which the Young-Stieltjes integral equation $f(x) = g(x) + \lambda \int_0^x K(x, y)\, df(y)$ has a solution $f(x)$ of bounded variation. (Received July 24, 1935.)

325. Mr. D. M. Dribin: *Representation of binary forms by sets of ternary forms.*

A. A. Albert has introduced the concept of the representation of integers by sets of forms and has studied the representation of positive integers by the set $\sum(d)$ of all positive classic ternaries of determinant d. O. K. Sagen, in his dissertation at Chicago, has treated a similar problem for non-classic ternaries. In the present paper, the representation of a given classic binary, $\phi = (a, t, b)$, $t^2 - ab \neq 0$, by a given set $\sum(d)$ of all classic ternaries of determinant d is studied. By a theorem of Gauss, the problem of representation of a given binary by $\sum(d)$ can be reduced to one of *proper* representation. Hence attention need be confined only to this latter case and in the present paper the problem of proper representation is completely solved. Of the results obtained the following theorem may be cited: A primitive binary ϕ of determinant

$\delta \neq 0$ is represented by a set $\sum(d)$ of all classic ternaries of determinant d if and only if $(-d\phi | p) = 1$ for every odd prime p dividing the kernel of δ and prime to d. (Received July 24, 1935.)

326. Professors A. H. Copeland and Francis Regan: *A postulational treatment of the Poisson law.*

A set of postulates relating to probability distributions for time series are presented in this paper. These postulates determine the character of a function $f(\alpha, E)$ where $f(\alpha, E)$ is the probability that there will be α points of the series within a given Lebesgue measurable set E. It is proved that, for a special case, this function has the classical form of the Poisson law. (Received July 26, 1935.)

327. Professors Virgil Snyder and Evelyn Carroll-Rusk: *The Veneroni transformation in S_n.*

The purpose of this paper is to obtain the set of bilinear equations in terms of which the transformation may be expressed, to obtain the complete set of multiple loci on the image of a general prime, and to map the lines, surfaces, and other varieties on straight lines, planes, three way spaces, and so on. A number of these properties have been obtained previously for S_4 but by means that could not be applied to the general case. (Received July 26, 1935.)

328. Dr. C. C. Torrance: *Paratingents in topological spaces.*

Point sets, called lines, having properties analogous to euclidean lines are defined in a non-metric topological space S by means of an extension of the concept of upper semi-continuous collection. Paratingents to point sets in S are defined by means of these lines after the manner of Bouligand. The theorem is proved that the paratingent to a point continuum C at a point of C is itself a line continuum. (Received July 26, 1935.)

329. Professor H. M. Gehman: *On extending a homeomorphism between two subsets of spheres.*

This paper is concerned with proving a theorem on the extension of a homeomorphism between two subsets of spheres similar to a theorem previously proved for subsets of planes (Transactions of this Society, vol. 31 (1929), pp. 241–252). The relationship between extension theorems for plane sets and for subsets of spheres is discussed. (Received July 29, 1935.)

330. Professor H. A. Davis and Dr. Amos Black: *A non-involutorial Cremona transformation belonging to the complex of secants of a twisted cubic.*

A transformation is defined by two pencils of quadric surfaces and a regulus of bisecants of the twisted cubic, all put into projective relation. The same transformation may be defined by pencils of cubic surfaces or nets of quartic surfaces. (Received July 31, 1935.)

331. Dr. L. A. Wolf: *Similarity of matrices in which the elements are real quaternions.*

This paper gives a necessary and sufficient condition that two matrices, A and B, of which the elements are real quaternions, be similar; that is, that there exist a non-singular matrix S whose elements are real quaternions such that $SAS^{-1}=B$. This paper defines a set of invariant factors for any matrix A, of which the elements are real quaternions, in terms of the ranks of certain real polynomials in A. The condition for similarity of two such matrices, A and B, is that the invariant factors of $(A-\lambda I)$ and $(B-\lambda I)$ be the same. (Received July 31, 1935.)

332. Professor Dunham Jackson: *Variations on Bernstein's theorem.*

In the theorem of Bernstein on the derivative of a polynomial in a finite interval the ends of the interval present a singularity which does not appear in the corresponding theorem on the derivative of a trigonometric sum for unrestricted real values of the variable. The author has shown (Transactions of this Society, vol. 26 (1924), pp. 139–145) that a proposition involving an analogous singularity holds for a trigonometric sum considered over a part of a period. The present paper gives a simplified proof of this result, and applies the same method to the derivation of a sequence of other theorems of similar character. (Received August 1, 1935.)

333. Professor A. J. Kempner: *Waring's problem and diophantine equations.*

In the author's thesis *Über das Waringsche Problem und einige Verallgemeinerungen* (Göttingen, 1912) it is noted that every problem of Waring type is completely equivalent to a certain linear diophantine equation with restrictions on the solutions. For example, "every positive integer = a sum of four or fewer squares," is equivalent to "the equation $n=1\cdot x_1+3\cdot x_2+5\cdot x_3+\cdots$ always has positive integral solutions with $4\geqq x_1\geqq x_2\geqq x_3\geqq\cdots$." On account of the growing importance of inequalities in many fields, and in view of the large number of Waring problems which have been solved, it seemed to the author worthwhile to collect the corresponding information available concerning diophantine equations. This is done in the present paper. (Received August 1, 1935.)

334. Professor A. J. Kempner: *Remarks on "unsolvability."*

This paper represents an effort of the author to make clear the mathematical and psychological difficulties involved in "unsolvability" problems. Two points have particularly attracted his attention. (a) It seems to the author that it is not frequently enough pointed out that there are essentially different ways in which a problem may be "unsolvable," and that this makes it unnecessarily hard for those not specializing in "Grundlagenforschung" to follow the literature. (b) Problems suggested by Brouwer and others suffer from the psychological handicap that one sees no mathematical reason why they should possibly be "unsolvable." The author attempts to outline the domain which, in discussions on unsolvability, even the strict classicist must concede to the intuitionist as unsettled. (Received August 1, 1935.)

335. Professor A. J. Kempner: *Anormal systems of numeration.*

The introduction of number systems to a base g not a positive integer leads to some interesting results and problems concerning uniqueness of representation, periodicity of decimal fractions, etc. These are considered in the present paper. (Received August 1, 1935.)

336. Professor Leonard Carlitz: *Note on higher congruences.*

In this paper are collected some results on congruences of the type $A_0t^{p^{nk}} + A_1t^{p^{n(k-1)}} + \cdots + A_kt \equiv B \pmod{P}$, where A, B, P are polynomials in an indeterminate x with coefficients in a Galois field of order p^n. The simplest case is that in which the associated operator $A_0E^k + \cdots + A_k$, where $Ey = y^{p^n}$ for all y, is factorable (mod P) into linear factors. Certain results are also obtained concerning the number of possible factorizations. (Received August 3, 1935.)

337. Professor W. G. Warnock: *Note on line configurations.*

In a previous paper (Tôhoku Mathematical Journal, vol. 36 (II) (1933), pp. 303–319) the author considered line configurations derived from the identity $p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0$ under group transformations. In this note attention is given to configurations formed by the intersections of these lines with the fundamental planes of the tetrahedron of reference of space S_3. In particular, a necessary and sufficient condition is given for the intersections to lie by pairs on rays through the vertices of the fundamental triangle of each face of the tetrahedron. (Received August 3, 1935.)

338. Mr. Aaron Herschfeld: *The equation* $2^x - 3^y = d$.

L. E. Dickson's *History of the Theory of Numbers* records that Leo Hebreus, or Levi Ben Gerson (1288–1344), proved that all powers of 2 and 3 differ by more than unity except the pairs 1 and 2, 2 and 3, 3 and 4, 8 and 9. This result was extended in 1918 by G. Pólya who showed that there can be only a finite number of solutions of the equation $a^x - b^y = d$, where a and b are any fixed positive integers and d is a given integer not equal to zero. In 1931 S. S. Pillai derived an asymptotic formula for the number of solutions of the inequality $0 < a^x - b^y \leqq n$. In the present paper the author finds all the solutions (x, y) of the inequality $|2^x - 3^y| \leqq 10$ by solving the twenty-one equations $2^x - 3^y = d$, $-10 \leqq d \leqq 10$. In addition it is proved that if $|d|$ is sufficiently great the equation $2^x - 3^y = d$ can have at most one solution. (Received July 12, 1935.)

339. Dr. C. B. Tompkins: *A note on continuous deformations locally one-to-one.*

In order to complete the results of an earlier paper, the author seeks a transformation which will untie knots in, but will not untwist, a homeomorph of an n-dimensional spherical shell in $(2n-1)$-space. Such a transformation is one which is the result of a continuous deformation locally one-to-one, defined in the following way. A transformation T transforming the variety X to Y is the result of a continuous deformation locally (1, 1) if there exists a set of transformations $S(\lambda)$ satisfying the conditions: (1) $S(\lambda)$ is a transformation defined uniquely for $0 \leqq \lambda \leqq 1$; (2) $S(0)=I$, the identity transformation; (3) $S(1)=T$; (4) the transform of each point of X as a function of λ is continuous; and (5) X may be covered by a finite number of neighborhoods of such a nature that on each neighborhood $S(\lambda)$ is a homeomorphism for all values of λ. Under such transformations all $(n-1)$-dimensional spheres with continuous derivatives are equivalent and every two topological shells with the same number of twists are equivalent. (Received August 2, 1935.)

340. Professor Leopold Fejér: *Trigonometric series and power series with repeatedly monotonic coefficients.*

Using previous results concerning the positive character of finite trigonometric sums or of the partial sums of higher order of some trigonometric series and the line of argument of G. Szegö, the author derives various facts concerning the positive and monotonic character of trigonometric series with terms in $\cos n\theta$, $\sin n\theta$, $\cos (2n-1)\theta$, or $\sin (2n-1)\theta$. Here the coefficients are real null sequences satisfying a monotonicity condition of a definite order. Under the same condition the univalence of power series is proved and estimates of the partial sums are given. For example: If $\{c_n\}$ is monotonic of order 4, the function $\sum c_n \cos n\theta$ is monotonically decreasing in $(0, \pi)$. Under the same condition $\sum c_n z^n$ is univalent in $|z|<1$. Further investigation has been devoted to the "remainders," $\sum_{v=0}^{\infty} c_v \sin (n+2v+1)\theta$. If $\{c_v\}$ is monotonic of order 3, this function has at least n zeros in $(0, \pi)$. Each of the intervals $(v-\frac{1}{2})\pi/n$, $v\pi/(n+1)$, $v \leqq [(n+1)/2]$, contains at least one of the zeros. This result gives a new and elementary proof of the estimates of A. Markoff and Stieltjes for the zeros of Legendre polynomials $P_n(\cos \theta)$. (Cf. a previous paper of G. Szegö.) In the present proof Heine's development of P_n in sines is used, and a new proof of this development is given. (Received August 16, 1935.)

341. Professor Dunham Jackson: *Some problems of closest approximations by means of trigonometric sums.*

In a recent abstract the writer has referred to some extensions of Bernstein's theorem, in which a characteristic feature of the demonstrations is the use of overlapping intervals covering the range of the independent variable, in connection with a transition from the case of polynomials to that of trigonometric sums. In the present paper the same device is used in proving theorems for normalized trigonometric sums and for trigonometric approximation analogous to those given for polynomials in the writer's paper on *Certain problems of closest approximation* (this Bulletin, vol. 39 (1933), pp. 889–906), together with some new results for the polynomial case. (Received August 22, 1935.)

342. Professor Harold Hotelling: *Relations between two sets of variates.*

Between two sets of normally correlated variates $x_1, \cdots, x_s, y_1, \cdots, y_t$

the correlations invariant under internal linear transformations are roots $\rho_1, \cdots, \rho_s$ (where $s \leqq t$) of a certain determinantal equation. The number of non-vanishing roots is the number of independent common components of the two sets. The primary statistical problem here is equivalent to the algebraic problem of finding invariants and standard forms for two quadratic forms in the x's and y's respectively and a bilinear form in the x's and y's, under transformations of the two sets separately. The solution of the algebraic problem gives the canonical forms $\sum_1^s x_i^2, \sum_1^t y_i^2, \sum_1^s \rho_i x_i y_i$. For statistical purposes it is also necessary to determine sampling distributions of these roots and of functions of them. Some of these sampling problems are solved completely in this paper, and approximations and partial solutions are given for others. In order to throw light on these sampling problems, contributions are made to the geometry of the curved four-dimensional manifold in 1–1 correspondence with the lines in 3-space, and of generalizations of this manifold. Certain problems of statistical inference involving a plurality of unknown parameters are discussed. Applications are numerous, including the problems of common components in mathematical psychology, anthropology, economics, meteorology, and other subjects. (Received August 7, 1935.)

343. Professor Gordon Pall: *On rational automorphs of binary quadratic forms.*

The denominator of any rational automorph of $f=[a, b, c]$ which carries an integral solution of $ax^2+bxy+cy^2=n$ into an integral solution must be a divisor of n. Any two integral solutions of this equation can be transformed into each other by proper (and improper) rational automorphs of f. (Received August 12, 1935.)

344. Dr. C. L. Siegel: *Mean values of arithmetic functions in number fields.*

Let $f(\xi)$ be a function defined for all integers of a totally real algebraic field $\mathfrak{K}$ of degree n. To ξ let correspond in the n-dimensional space the point whose coordinates are the n conjugate numbers to ξ. For any domain G in that space we form the sum $F=\sum_{\xi \text{ in } G} f(\xi)$, in which ξ assumes all integral values in $\mathfrak{K}$ for which the associated point ξ is situated in G. In some investigations of number theory an asymptotic expression of F, if G becomes in certain ways infinitely large, is sought. The content of this paper is a formula which generalizes a well known formula for the sum of the coefficients of a Dirichlet series. An application to the problem of divisors is given. (Received August 14, 1935.)

345. Dr. C. L. Siegel: *The volume of the fundamental domain for some infinite groups.*

The purpose of the paper is the proof of a formula which gives an explicit expression for the volume of fundamental domains for certain classes of infinite groups. A special application gives the non-euclidean volume of the fundamental domain for the modular group in every totally real algebraic field. Blumenthal and Hecke have shown the importance of the corresponding modular functions for algebraic arithmetic investigations. Since the knowledge of a set of generators of the modular group is necessary for the construction of any example in the theory of modular functions, the determination of the volume of the fundamental domain can be useful for further researches. (Received August 14, 1935.)

346. Dr. C. L. Siegel: *On the algebraic integrals of the restricted three-body problem.*

Among the few general recent results in the problem of three bodies the theorem of Bruns deserves special interest in spite of its negative character. The statement is that the ten well known integrals constitute the complete field of algebraic integrals of the problem. The present paper proves an analogous result for the restricted problem of three bodies. The result cannot be obtained by specializing Bruns' theorem. (Received August 14, 1935.)

347. Mr. Benjamin Rosenbaum: *Generalizations of some theorems of Schur on irreducible polynomials.*

In this paper it is shown that polynomials $f(x)=1+\sum_{v=1}^{n-1}g_v a!x^{lv}/(kv-b)!\pm a!x^{ln}/(kn-b)!$, as well as polynomials $g(x)$ which are given by the formula $g(x)=1+\sum_{v=1}^{n-1}g_v u_{2a}x^{lv}/u_{2(kn-b)}\pm u_{2a}x^{ln}/u_{2(kn-b)}$, where $u_{2j}=1\cdot 3\cdot 5\cdots(2j-1)$ and where g_v, a, b, k, l, n are rational integers with certain restrictions, are irreducible in the rational field. The restrictions are: $1\leqq a\leqq(kn-b)/2$; $0\leqq b\leqq kn-2$; $k, l\geqq 1$; $n>1$. When $kv-b<1$, replace it by 1. This is a generalization of the results of I. Schur (Berliner Sitzungsberichte (1929), pp. 125, 370). It is further shown that $f(x)$ and $g(x)$ are special cases of an irreducible polynomial defined by the divisibility properties of the coefficients. For certain values of $kn-b$ the irreducibility of $f(x)$ and $g(x)$ can be established by means of the Eisenstein and Koenigsberger criteria. (Received August 22, 1935.)

348. Dr. L. M. Blumenthal (National Research Fellow): *Isometric geometry methods in determinant theory.*

Determinants of the type $|r_{ij}|$, $r_{ij}=r_{ji}>0$, $(i\neq j)$, $r_{ii}=0$, $r_{0i}=1$, $(i\neq 0)$, $(i, j=0, 1, \cdots, m)$, and of type $|r_{ij}|$, $r_{ij}=r_{ji}$, $r_{ii}=1$, $(i, j=1, 2, \cdots, m)$, are studied in this paper, and new theorems concerning them are presented. These theorems are obtained by applying results in the metric characterization of n-dimensional euclidean, spherical, and hyperbolic spaces. (Received August 24, 1935.)

349. Mr. I. E. Perlin: *Sufficient conditions for a minimum in the problem of Lagrange with isoperimetric conditions.*

In this paper sufficient conditions are established for a minimum in the parametric problem of Lagrange with isoperimetric side conditions. These conditions can be deduced directly from those for an associated non-parametric Lagrange problem without isoperimetric side conditions. These results are established by the aid of a generalized Lindeberg theorem which is developed in this paper. Two generalized Osgood theorems applicable to the parametric problem of Lagrange with isoperimetric side conditions are developed. The methods and results employed in this paper are those of Tonelli and McShane. No use whatsoever is made of the notion of a field. (Received September 3, 1935.)

350. Dr. C. W. Vickery: *Concerning spaces of uncountably many dimensions.*

The author has shown that every $\aleph_\alpha$-separable metric space is homeomorphic with a subset of space D_{ω_α} and that every $\aleph_\alpha$-completely separable, normal, topological space is homeomorphic with a subset of space E_{ω_α}. Thus in order that a space be metric it is necessary and sufficient that it be homeomorphic with a subset of some space D_{ω_α}. Spaces D_{ω_α} and E_{ω_α} were defined by the author in a paper presented to the Society recently. (Received September 3, 1935.)

351. Professor N. H. McCoy: *On the characteristic roots of matric polynomials.*

Let $A_1, A_2, \cdots, A_m$ be finite matrices of order n, and P the algebra of polynomials in these matrices with coefficients in the field of complex numbers. The following three properties are shown to be equivalent: (1) the characteristic roots of every polynomial $f(A_1, A_2, \cdots, A_m)$ are all of the form $f(\lambda_1, \lambda_2, \cdots, \lambda_m)$, where λ_i is a characteristic root of A_i $(i=1, 2, \cdots, m)$; (2) every matrix $A_iA_j-A_jA_i$ $(i\neq j)$ is zero or properly nilpotent in P; (3) there exists a non-singular matrix R such that each matrix RA_iR^{-1} is in triangle form. Special cases of one or more of these properties have been studied by Bruton, Ingraham, Roth (this Bulletin, abstracts 38-9-196, 38-9-197, 41-3-168, respectively), and Williamson (American Journal of Mathematics, vol. 57 (1935), pp. 281–293). Direct generalizations of the results mentioned are also obtained for the case in which it is required that just $r(\leqq n)$ of the characteristic roots of every polynomial be of the form stated in (1). The proofs are based on theorems of E. Noether on the representations of algebras. (Received September 5, 1935.)

352. Dr. Max Zorn: *Discontinuous groups in topological spaces.* Preliminary report.

For groups G, discontinuous (in the sense of Baer-Levi) in a connected, locally connected, separable, locally compact space T, there exists a fundamental domain F with the following important property: If a set S of points in the closure $\bar{F}$ of F, plus all its images under G, has a limit point, then S has already a limit point. If $\bar{F}$ is compact, G has a finite number of generators. If T is a locally euclidean space, T and $\bar{F}$ may be considered as complexes; as a special result we obtain the characterization (Sperner, Kerékjártó) of plane translations. The following application to hypercomplex arithmetics is also discussed: the units of an order in a semi-simple algebra over the rational numbers form a group, which has a finite number of generating elements with a finite number of defining relations. (Received September 5, 1935.)

353. Dr. Max Zorn: *On Schur's generalization of Fermat's theorem.*

Schur defined the derivative of the sequence $a_0, a_1, \cdots, a_n$ with respect to the rational prime p as the sequence of quotients $a_{n+1}-a_n/p^{n+1}$. The derivative of the sequence $1, a, a^p, \cdots$, where $a_{n+1}=a_n{}^p$, is integral (Fermat). Schur proved the first $p-1$ derivatives to be integral for $(a, p)=1$, and got some further results. In this paper the meaning of Schur's theorems in p-adic analysis is given. A simpler proof is obtained in terms of p-adic logarithms and exponential functions. This proof gives in addition the congruence value mod p of numbers which are integers after Schur. (Received September 5, 1935.)

354. Professor J. W. Lasley: *On Monge's differential equation.*

This paper presents a solution of a differential equation of Monge by means of the theory of linear dependence. Since the differential equation in question has an historical setting of uncommon interest, the salient facts are noted. (Received September 5, 1935.)

355. Mr. Garrett Birkhoff: *A new definition of limit.*

Let H be any Hausdorff space. An aggregate Σ of sets of points of H will be said to *converge* to a point p of H if and only if (1) the set-theoretical product of any two sets of Σ contains a suitable set of Σ, and (2) any neighborhood of p contains a suitable set of Σ. By modifying this definition one can (1) construct a topological group out of the homeomorphisms of any Hausdorff space with itself; (2) complete any topological group or topological linear space; and (3) complete any *multiply distanced space.* (Received September 5, 1935.)