$$
\begin{aligned}
&[|a_{p_i}| - \alpha^{q_i}] - \left[\frac{k^{p_i}}{1-k} + |\epsilon_{p_i}||a_{p_i}| + \epsilon'_{p_i}\frac{\beta^{p_i+1}}{1-\beta}\right] \\
&\quad = |a_{p_i}|(1 - |\epsilon_{p_i}|) - \alpha^{q_i} - \frac{k^{p_i}}{1-k} - \epsilon'_{p_i}\frac{\beta^{p_i+1}}{1-\beta} \\
&\quad > \beta^{p_i}(1 - |\epsilon_{p_i}|) - \alpha^{q_i} - \frac{k^{p_i}}{1-k} - \epsilon'_{p_i}\frac{\beta^{p_i+1}}{1-\beta} \\
&\quad = \beta^{p_i}\left(1 - |\epsilon_{p_i}| - \frac{\alpha^{q_i}}{\beta^{p_i}} - \frac{k^{p_i}}{\beta^{p_i}(1-k)} - \epsilon'_{p_i}\frac{\beta}{1-\beta}\right).
\end{aligned}
$$

Now for i sufficiently large all the terms within the last parentheses except the first are as small as we please. Hence for sufficiently large i the difference in question is positive. From this contradiction the theorem follows.

In conclusion, we may note as a simple corollary of the above theorem that if $\overline{\lim}_{n\to\infty}|a_n|^{1/n}=1$, then $\lim_{n\to\infty}|a_{n+1}/a_n|=1$ if and only if there exists a sequence of *real* numbers λ_n such that $\lim_{n\to\infty}\lambda_n=1$ and $\overline{\lim}_{n\to\infty}\big|\,|a_{n+1}|-\lambda_n|a_n|\,\big|^{1/n}<1$.

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ON THE COEFFICIENTS OF A TYPICALLY-REAL FUNCTION*

BY M. S. ROBERTSON†

1. *Introduction.* It is well known‡ that if

$$
f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1}
$$

is regular for $|z| \leqq 1$, and if E is defined by the formula

$$
E \equiv \underset{|z_1|=|z_2|=1}{\text{maximum}} \left|\mathcal{R}f(z_1) - \mathcal{R}f(z_2)\right|, \tag{2}
$$

* Presented to the Society, February 23, 1935.

† National Research Fellow.

‡ See E. Landau, Archiv der Mathematik und Physik, (3), vol. 11 (1906), pp. 31–36.

or, in other words, if E is the oscillation of the real part of $f(z)$ for all points z_1 and z_2 on the unit circle, then

$$|a_1| = |f'(0)| \leqq \frac{2}{\pi} E. \tag{3}$$

In this paper an analogous result and extensions are obtained for *all* the coefficients of any function $f(z)$ regular and *typically-real* for $|z| < 1$.

DEFINITION. A function $f(z)$, $f(0)=0$, $f'(0)\neq 0$, regular for $|z| < R$, is said to be typically-real with respect to the circle $|z| = R$, if within this circle $f(z)$ is real for, and only for, the points on the real axis.*

It may be noticed, as W. Rogosinski has pointed out, that the class of functions regular and univalent in the circle $|z| = R$ and real on the real axis form a subclass of the class of functions typically-real with respect to this circle.

2. *A Stieltjes Integral Representation for Typically-Real Functions.* Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (a_n \text{ real}), \tag{4}$$

be regular and typically-real for $|z| < 1$. Then it is known† that $f(z)$ can be represented in the form

$$f(z) = \frac{zg(z)}{1 - z^2}, \tag{5}$$

where $g(z)$ is regular for $|z| < 1$, $g(0)=1$, $\mathfrak{R}g(z) > 0$ for $|z| < 1$. Further, by the formula of G. Herglotz,‡ we may write

$$\begin{aligned} g(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 + e^{-i\theta}z}{1 - e^{-i\theta}z}\, d\alpha(\theta) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - z^2 - 2iz\sin\theta}{1 - 2z\cos\theta + z^2}\, d\alpha(\theta), \end{aligned} \tag{6}$$

* See W. Rogosinski, *Über positive harmonische Entwicklungen und typisch-reelle Potenzreihen*, Mathematische Zeitschrift, vol. 35 (1932), pp. 93–121.

† See W. Rogosinski, loc. cit., p. 99.

‡ See G. Herglotz, Leipziger Berichte, 1911, pp. 501–511.

where $\alpha(\theta)$ is an odd non-decreasing function of θ in the interval $(-\pi, \pi)$, and where $f(z)$ is real on the real axis, as is also $g(z)$. Hence

$$
(7)\qquad \begin{aligned} f(z) &= \frac{1}{\pi}\int_0^{\pi} \frac{z\,d\alpha(\theta)}{1 - 2z\cos\theta + z^2} \\ &= \frac{1}{\pi}\int_0^{\pi}\left(\sum_{n=1}^{\infty} \frac{\sin n\theta}{\sin\theta} z^n\right) d\alpha(\theta). \end{aligned}
$$

L. Fejér has observed* that

$$
(8)\qquad F(z) = \int_0^z \frac{f(z)}{z}\,dz = z + \sum_2^{\infty} \frac{a_n}{n} z^n,
$$

is univalent and convex in the direction of the imaginary axis for $|z| < 1$, that is, no straight line parallel to the imaginary axis can cut the image of the circle $|z| = r$ (for every r in the interval $0 < r < 1$) mapped by the function $F(z)$ in more than two points. It follows from (7) and (8) by integration that

$$
(9)\qquad F(z) = \frac{1}{2\pi i}\int_0^{\pi} \log\left\{\frac{1 - ze^{-i\theta}}{1 - ze^{i\theta}}\right\} \frac{d\alpha(\theta)}{\sin\theta}.
$$

3. *The Coefficients of a Typically-Real Function.* From (9), since $F(r)$ is real, we have

$$
(10)\qquad F(r) = \frac{1}{\pi}\int_0^{\pi} \arg\,(1 - re^{-i\theta}) \frac{d\alpha(\theta)}{\sin\theta}.
$$

Since the integrand is an increasing function of r for every θ, we have

$$
(11)\qquad \begin{aligned} F(1) \equiv \lim_{r\to 1} F(r) &= \frac{1}{\pi}\int_0^{\pi} \lim_{r\to 1} \arg\,(1 - re^{-i\theta}) \frac{d\alpha(\theta)}{\sin\theta} \\ &= \frac{1}{2\pi}\int_0^{\pi} \frac{\pi - \theta}{\sin\theta}\,d\alpha(\theta). \end{aligned}
$$

Similarly we also have

$$
(12)\qquad F(-1) \equiv \lim_{r\to 1} F(-r) = \frac{-1}{2\pi}\int_0^{\pi} \frac{\theta}{\sin\theta}\,d\alpha(\theta).
$$

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* See L. Fejér, Journal of the London Mathematical Society, vol. 8 (1933), p. 61, footnote.

These limits are finite or infinite according as the integrals exist or do not exist. Hence if we define

$$(13)\qquad E(r) \equiv F(r) - F(-r) = \int_{-r}^{r} \frac{f(t)}{t}\,dt = \int_{-1}^{+1} \frac{f(rt)}{t}\,dt,$$

then

$$(14)\qquad E \equiv \lim_{r\to 1} E(r) = \frac{1}{2}\int_0^{\pi} \frac{d\alpha(\theta)}{\sin\theta},$$

and is finite whenever this integral exists. However, since $F(z)$ is convex in the direction of the imaginary axis, and since $E(r)$ is the length of the segment of the real axis intercepted by the contour into which $|z| = r$ is mapped by $F(z)$, we have

$$(15)\qquad |\mathfrak{R}F(z_1) - \mathfrak{R}F(z_2)| \leqq F(r) - F(-r) = E(r)$$

for all z_1 and z_2 on $|z| = r$. Thus $E(r)$ denotes the oscillation of the real part of $F(z)$ on $|z| = r$.

From (4) and (7) we have, by comparing coefficients on both sides of the equation (7),

$$(16)\qquad a_n = \frac{1}{\pi}\int_0^{\pi} \frac{\sin n\theta}{\sin\theta}\,d\alpha(\theta),$$

$$(17)\qquad |a_n| \leqq \frac{1}{\pi}\int_0^{\pi} \left|\frac{\sin n\theta}{\sin\theta}\right| d\alpha(\theta) \leqq \frac{1}{\pi}\int_0^{\pi} \frac{d\alpha(\theta)}{\sin\theta}.$$

Whenever E is finite we have, by (14) and (17),

$$(18)\qquad |a_n| \leqq \frac{2}{\pi}E, \qquad \text{for all } n,$$

$$(19)\qquad \frac{1}{n+1}\sum_{k=1}^{n} |a_k| \leqq \frac{1}{\pi}\int_0^{\pi} M_n(\theta)\,\frac{d\alpha(\theta)}{\sin\theta},$$

where

$$M_n(\theta) \equiv \frac{1}{n+1}\sum_{k=1}^{n} |\sin n\theta|.$$

However, as T. Gronwall has shown,*

* See T. Gronwall, Transactions of this Society, vol. 13 (1912), pp. 445–468.

$$M_n(\theta) < \sin z_0 = 0.72457 \cdots, \tag{20}$$

where z_0 is the positive root of the equation $\tan(z_0/2) = z_0$. Further,

$$\lim_{n\to\infty} M_n(\theta) = M(\theta) \leqq \frac{2}{\pi}, \tag{21}$$

$$\begin{aligned} \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{k=1}^{n} |a_k| &\leqq \frac{1}{\pi}\int_0^{\pi} \lim_{n\to\infty} M_n(\theta)\,\frac{d\alpha(\theta)}{\sin\theta} \\ &\leqq \frac{2}{\pi}\,\frac{1}{\pi}\int_0^{\pi} \frac{d\alpha(\theta)}{\sin\theta} \leqq \left(\frac{2}{\pi}\right)^2 E. \end{aligned} \tag{22}$$

Similarly,

$$\frac{1}{n+1}\sum_{k=1}^{n} |a_k| < \left(\frac{2\sin z_0}{\pi}\right) E < \left(\frac{1.45}{\pi}\right) E, \tag{23}$$

for all n.

Again, if we denote by Γ_n the expression

$$\Gamma_n = \max_{\theta} \sum_{k=1}^{n} \frac{|\sin k\theta|}{k}, \tag{24}$$

then we have*

$$\frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{k} < \Gamma_n < \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{k} + \frac{2}{\pi}, \tag{25}$$

$$\lim_{n\to\infty} \frac{\Gamma_n}{\log n} = \frac{2}{\pi}. \tag{26}$$

Hence by the method used above in (19) we may show that

$$\sum_{k=1}^{n} \frac{|a_k|}{k} < \left\{\left(\frac{2}{\pi}\right)^2 + \left(\frac{2}{\pi}\right)^2 \sum_{k=1}^{n} \frac{1}{k}\right\} E, \tag{27}$$

$$\overline{\lim_{n\to\infty}} \frac{1}{\log n} \cdot \sum_{k=1}^{n} \frac{|a_k|}{k} \leqq \left(\frac{2}{\pi}\right)^2 E. \tag{28}$$

Let

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* See G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, 1925, pp. 81 and 274.

$$A(n, \theta) \equiv \sum_{k=1}^{n} \frac{\sin k\theta}{k}. \tag{29}$$

Then the absolute maximum* of $A(n, \theta)$ is $A(n, \pi/(n+1))$. Consequently we obtain by the above method

$$\left| \sum_{k=1}^{n} \frac{a_k}{k} \right| \leqq \frac{2}{\pi} A\left(n, \frac{\pi}{n+1}\right) E. \tag{30}$$

For n odd the factor $2/\pi$ in (18) cannot be replaced by a smaller one, since for the function $f(z) = z(1+z^2)^{-1}$ we have

$$| a_{2n-1} | = 1, \quad a_{2n} = 0, \quad F(z) = \text{arc tan } z, \quad E = \frac{\pi}{2}.$$

Hence equality is attained by $z(1+z^2)^{-1}$ for every odd value of n. However, one cannot have equality for all n, even and odd, for a given function of the class under consideration, as this would contradict the inequality (22).

4. *A Class of Odd Typically-Real Functions.* Let I denote the class of odd functions

$$f(z) = z + \sum_{n=1}^{\infty} b_{2n+1} z^{2n+1} \tag{31}$$

with the properties

(a) $f(z)$ is regular for $|z| < 1$,

(b) $f(z)$ is real on the real axis, that is, b_{2n+1} is real for all n,

(c) $f(z)$ lies inside the jth quadrant whenever z is inside the jth quadrant for $|z| < 1$, $(j = 1, 2, 3, 4)$.

The class of odd functions regular and univalent for $|z| < 1$ and real on the real axis form a subclass of I.

THEOREM. *If*

$$f(z) = z + \sum_{n=1}^{\infty} b_{2n+1} z^{2n+1}$$

belongs to class I, then

$$| b_3 | \leqq 1, \quad | b_{2n-1} | + | b_{2n+1} | \leqq 2, \quad \sum_{k=1}^{n} \frac{b_{2k+1}}{k} \geqq -2.$$

* See G. Pólya and G. Szegö, loc. cit., p. 79.

PROOF. Let $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$.

$$(32) \qquad b_{2n+1}r^{2n+1} = \frac{1}{\pi}\int_{-\pi}^{\pi} v(r, \theta) \sin (2n+1)\theta \, d\theta,$$

$$(33) \qquad b_{2n+1}r^{2n+1} = \frac{1}{\pi}\int_{-\pi}^{\pi} u(r, \theta) \cos (2n+1)\theta \, d\theta.$$

On account of hypotheses (b) and (c) of the definition of class I we may write

$$(34) \qquad b_{2n+1}r^{2n+1} = \frac{4}{\pi}\int_{0}^{\pi/2} v(r, \theta) \sin (2n+1)\theta \, d\theta,$$

$$(35) \qquad b_{2n+1}r^{2n+1} = \frac{4}{\pi}\int_{0}^{\pi/2} u(r, \theta) \cos (2n+1)\theta \, d\theta,$$

where $v(r, \theta) > 0$, $u(r, \theta) > 0$ for $0 < \theta < \pi/2$, $r < 1$. From (34) we have

$$(36) \qquad b_{2n+1}r^{2n+1} - b_{2n-1}r^{2n-1} = \frac{8}{\pi}\int_{0}^{\pi/2} v(r, \theta) \cos 2n\theta \sin \theta \, d\theta,$$

so that

$$\begin{aligned} \left| b_{2n+1}r^{2n+1} - b_{2n-1}r^{2n-1} \right| &\leqq \frac{8}{\pi}\int_{0}^{\pi/2} \left| v(r, \theta) \sin \theta \right| d\theta \\ &\leqq \frac{8}{\pi}\int_{0}^{\pi/2} v(r, \theta) \sin \theta \, d\theta \\ &\leqq 2r. \end{aligned}$$

Letting $r \to 1$, we have

$$(37) \qquad \left| b_{2n-1} - b_{2n+1} \right| \leqq 2.$$

From (35) we have similarly

$$(38) \qquad \begin{aligned} \left| b_{2n-1}r^{2n-1} + b_{2n+1}r^{2n+1} \right| &\leqq \frac{8}{\pi}\int_{0}^{\pi/2} \left| u(r, \theta) \cos 2n\theta \cos \theta \right| d\theta \\ &\leqq \frac{8}{\pi}\int_{0}^{\pi/2} u(r, \theta) \cos \theta \, d\theta \\ &\leqq 2r. \end{aligned}$$

Letting $r \to 1$ again, we have

$$|b_{2n-1} + b_{2n+1}| \leqq 2. \tag{39}$$

Combining (37) and (39) we obtain

$$|b_{2n-1}| + |b_{2n+1}| \leqq 2, \qquad \text{(for all } n). \tag{40}$$

The inequalities (40) were established by a different method for the subclass of I consisting of odd univalent functions real on the real axis by J. Dieudonné.* Further, since we have†

$$B(n, \theta) \equiv \sum_{k=1}^{n} \frac{\cos k\theta}{k} \geqq -1 \tag{41}$$

for all n, then

$$\begin{aligned}\sum_{k=1}^{n} & \frac{(b_{2k+1}r^{2k+1} - b_{2k-1}r^{2k-1})}{k} \\ &= \frac{8}{\pi}\int_0^{\pi/2} B(n, \theta)v(r, \theta) \sin\theta \, d\theta \geqq -2r,\end{aligned} \tag{42}$$

$$\sum_{k=1}^{n} \frac{(b_{2k+1} - b_{2k-1})}{k} \geqq -2, \tag{43}$$

and similarly

$$\sum_{k=1}^{n} \frac{(b_{2k+1} + b_{2k-1})}{k} \geqq -2. \tag{44}$$

On adding (43) and (44) we obtain also

$$\sum_{k=1}^{n} \frac{b_{2k+1}}{k} \geqq -2. \tag{45}$$

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* See J. Dieudonné, Annales de l'École Normale, vol. 48 (1931), p. 318.

† See G. Pólya und G. Szegö, loc. cit., p. 79.