SPACE INVOLUTORIAL TRANSFORMATIONS OF THE GEISER AND BERTINI TYPES*

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1. *Introduction.* One form of generalization of a plane involution is a space involutorial transformation in which each plane of a pencil is invariant and in each such plane there is a plane involution of the same type. Particular examples of this for Geiser and Bertini involutions have been given by Carroll,† Snyder and Lehr,‡ and Sharpe and Dye.§ I shall discuss a more general form for space involutorial transformations arising from these plane involutions by means of a mapping on a cubic surface.|| The Bertini transformation obtained has the signature

$$I_{120n+51}: l^{120n+34+6t} + (O, \bar{O})^{120n+40} + C^{6}_{12n+6};$$

the Geiser transformation has the signature

$$I_{24n+19}: l^{24n+11+3t} + O^{24n+14} + C^{3}_{12n+6}.$$

2. *The Geiser Transformation.* In an involutorial space transformation of the Geiser type, let $x_4 = \lambda x_3$ be the equation of the invariant pencil of planes, and let I_G be the Geiser involution in the plane $x_4 = \lambda x_3$. Choose one of the fundamental points of I_G as $O \equiv (1, \lambda, 0, 0)$ on the line $l \equiv x_3 = x_4 = 0$, and map the I_G on a cubic surface F_3 by means of the bilinear $T_{3\text{-}3}$ defined by the matrices

(1) $$\left\|(a_{i1}y_i)\ (a_{i2}y_i)\ (a_{i3}y_i)\ (a_{i4}y_i)\right\|,$$

(2) $$\left\|(a_{1i}x_i)\ (a_{2i}x_i)\ (a_{3i}x_i)\ (a_{4i}x_i)\right\|,$$

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† E. T. Carroll, American Journal of Mathematics, vol. 54 (1931), pp. 707–717, and vol. 56 (1934), pp. 96–108.

‡ V. Snyder and M. Lehr, American Journal of Mathematics, vol. 53 (1931), pp. 186–195.

§ F. R. Sharpe and L. A. Dye, Transactions of this Society, vol. 36 (1934), pp. 292–305.

|| For a discussion of the mapping of a Geiser or a Bertini plane involution on a cubic surface, see H. F. Baker, *Principles of Geometry*, vol. 6, pp. 122–130.

where here and in the rest of the paper only the first row of determinants and matrices is written; the second and third rows are obtained by replacing a_{ij} by b_{ij} and c_{ij}, respectively. To the plane $x_4=\lambda x_3$ corresponds the surface

$$(3) \qquad F_3 = \left| (a_{i1}y_i) \ (a_{i2}y_i) \ (a_{i3}' y_i) \right| = 0,$$

where $a'_{i3}=a_{i3}+\lambda a_{i4}$. The point O goes into the point $P\equiv(D_i)$, where the D_i are the third-order determinants of the matrix $\|a'_{11} \ a'_{21} \ a'_{31} \ a'_{41}\|$, and where $a'_{i1}=a_{i1}+\lambda a_{i2}$. The image of the line l is the cubic curve $C_3\equiv\|(a_{i1}y_i) \ (a_{i2}y_i)\|=0$. The pairs of corresponding points of the Geiser involution J_G on the surface F_3 are determined by the pairs of intersections with F_3 of lines through P.

In the J_G on F_3 the image of P is the cubic curve cut from F_3 by the tangent plane p at P. The equation of p is

$$(4) \qquad \left| (a'_{i1}y_i) \ (a_{i2}D_i) \ (a'_{i3}D_i) \right| = 0.$$

The quadric cone H through C_3 with vertex at P has the equation

$$(5) \qquad \left| (a'_{i1}y_i) \ (a_{i2}y_i) \ (a_{i2}D_i) \right| = 0,$$

and meets F_3 in a residual cubic which is the image of C_3. The invariant curve of the J_G is cut from F_3 by the polar quadric of P with respect to F_3. This quadric K has for its equation

$$(6) \qquad \left| (a'_{i1}y_i) \ (a_{i2}D_i) \ (a'_{i3}y_i) \right| + \left| (a'_{i1}y_i) \ (a_{i2}y_i) \ (a'_{i3}D_i) \right| = 0.$$

If a_{ij}, b_{ij}, and c_{ij} are polynomials of order n in λ, and if λ is replaced by x_4/x_3, then the surfaces $y_i=0$ in the (x) space are of order $3n+3$. These surfaces have l as a $3n$-fold line and contain a curve C_{12n+6} of order $12n+6$, of genus $24n+3$, which meets l in $12n$ points. Since any plane through l is invariant under the Geiser space transformation, there is a pencil of homaloidal surfaces consisting of the pencil of planes through l and the images of l and O. The equations of these image surfaces are obtained by replacing y_i in the equations of H and p by the third-order determinants of the matrix (2). Since H, p, and K are of orders $12n+3$, $12n+7$, and $12n+4$ in λ, respectively, they correspond to surfaces of orders $12n+8$, $12n+10$, and $12n+10$.

The table of characteristics of the space involutorial transformation I_{24n+19} can now be written:

$$
\begin{aligned}
O &\sim F_{12n+10}: l^{12n+7+2t} + O^{12n+9} + C_{12n+6},\\
l &\sim F_{12n+8}: l^{12n+3+t} + O^{12n+4} + C^{2}_{12n+6},\\
C_{12n+6} &\sim F_{60n+44}: l^{60n+26+6t} + O^{60n+32} + C^{7}_{12n+6},\\
S_1 &\sim S_{24n+19}: l^{24n+11+3t} + O^{24n+14} + C^{3}_{12n+6},\\
&\quad K_{12n+10}: l^{12n+4+2t} + O^{12n+6} + C^{2}_{12n+6},
\end{aligned}
$$

where the coefficient of t indicates the number of fixed tangent planes at a point of l. The image of the C_{12n+6} is obtained by applying the transformation to an S_{24n+19}.

The parasitic lines of the transformation consist of the trisecants of C_{12n+6} which meet l, and the bisecants of C_{12n+6} which pass through O. The number of trisecants of a C_m which meet a line l having i intersections with C_m is*

$$(m - 2)[h - m(m - 1)/6] - i(h - m + 2) + i(i - 1)(i - 2)/6,$$

where h is the number of apparent double points of C_m. The number of trisecants of C_{12n+6} which meet l is $24n+8$. In order to obtain the number of bisecants through O, it is necessary to set up a correspondence. Given a point μ on the line l, there are $h' = h - 12n(12n-1)/2 = 36n+7$ bisecants through it which determine h' planes λ. Given a plane λ through l there are six intersections of C_{12n+6} with it not on l and hence fifteen bisecants which determine fifteen points μ. In the (λ, μ) correspondence there are $h'+15 = 36n+22$ coincidences. There are then $36n+22$ bisecants of C_{12n+6} which meet O, and the total number of parasitic lines is $60n+30$.

Let ζ be the number of parasitic conics of the transformation, and let η be the number of parasitic cubics. The complete intersection of two surfaces of the web of S_{24n+19} is made up of

$$
\begin{aligned}
(24n + 19)^2 = 24n + 19 + (24n + 11)^2 + 9 + 9(12n + 6)\\
+ 60n + 30 + 8\zeta + 27\eta
\end{aligned}
$$

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* L. A. Dye, this Bulletin, vol. 41 (1935), pp. 109–110.

curves, and the complete intersection of an S_{24n+19} and the K_{12n+10} is made up of

$$(24n + 19)(12n + 10) = 12n + 10 + (12n + 4)(24n + 11) + 6 + 6(12n + 6) + 60n + 30 + 4\zeta + 9\eta$$

curves. The solution of these equations is $\zeta = 24n+16$, $\eta = 0$; therefore there are $24n+16$ conics and $60n+30$ lines of the second species in the I_{24n+19}.

3. *The Bertini Transformation.* The methods of the last section are now used to study an involutorial space transformation of the Bertini type. Let the invariant pencil of planes have the line $l \equiv x_3 = x_4 = 0$ as axis. In a plane $x_4 = \lambda x_3$ let $O \equiv (1, \mu, 0, 0)$ and $\overline{O} \equiv (1, -\mu, 0, 0)$, $(\mu^2 = \lambda)$, be two fundamental points of the Bertini involution I_B in the plane. The I_B is mapped on the cubic surface (3) by means of the $T_{3\text{-}3}$ defined by the matrices (1) and (2). The images of O and $\overline{O}$ are $P \equiv (D_i)$ and $\overline{P} \equiv (\overline{D}_i)$; D_i and $\overline{D}_i$ are the third-order determinants of the matrices $\|a'_{11}\ a'_{21}\ a'_{31}\ a'_{41}\|$, $\|\bar{a}'_{11}\ \bar{a}'_{21}\ \bar{a}'_{31}\ \bar{a}'_{41}\|$, where $a'_{i1} = a_{i1} + \mu a_{i2}$ and $\bar{a}_{i1} = a_{i1} - \mu a_{i2}$. The image of l is the C_3 given by the matrix equation $\|(a_{i1}y_i)\ (a_{i2}y_i)\| = 0$.

The tangent planes to F_3 at P, $\overline{P}$ are p, $\bar{p}$ and they have as equations

$$p \equiv |\,(a'_{i1}\,y_i)(\bar{a}'_{i1}\,D_i)\,(a'_{i3}D_i)\,| = 0,$$
$$\bar{p} \equiv |\,(a'_{i1}\,\overline{D}_i)(\bar{a}'_{i1}\,y_i)(a'_{i3}\,\overline{D}_i)\,| = 0.$$

We now define two numbers d, $\bar{d}$ as follows:

$$d \equiv p(\overline{D}_i) \equiv |\,(a'_{i1}\,\overline{D}_i)\,(\bar{a}'_{i1}D_i)\,(a'_{i3}D_i)\,|,$$
$$\bar{d} \equiv \bar{p}(D_i) \equiv |\,(a'_{i1}\,\overline{D}_i)\,(\bar{a}'_{i1}D_i)\,(a'_{i3}\overline{D}_i)\,|.$$

The residual intersection R of the line $P\overline{P}$ with F_3 has as coordinates $(\bar{d}D_i - d\overline{D}_i)$. The equation of the tangent plane to F_3 at R is

$$r \equiv d^2\bar{p} + \bar{d}^2 p - d\bar{d}\{\,|\,(a'_{i1}\,y_i)\,(\bar{a}'_{i1}D_i)\,(a'_{i3}\overline{D}_i)\,| + |\,(a'_{i1}\overline{D}_i)\,(\bar{a}'_{i1}\,y_i)\,(a'_{i3}D_i)\,| + |(a'_{i1}\overline{D}_i)\,(a'_{i1}D_i)(a'_{i3}\,y_i)\,|\,\} = 0.$$

The pairs of corresponding points in the Bertini involution J_B on F_3 are cut out by the conics tangent to F_3 at P and $\overline{P}$.

The image of P in J_B is the sextic cut from F_3 by the quadric having contact with F_3 at $\overline{P}$ and contact of the second order at P. Its equation is

$$\begin{aligned} H \equiv \bar{d}^2 d\big[\,\big|\, (a'_{i1}y_i)(\bar{a}'_{i1}D_i)(a'_{i3}y_i) \,\big| \\ + \big|\, (a'_{i1}y_i)(\bar{a}'_{i1}y_i)(a'_{i3}D_i) \,\big|\, \big] + pr = 0; \end{aligned}$$

similarly the quadric which determines the image of $\overline{P}$ has the equation

$$\begin{aligned} \overline{H} \equiv d^2\bar{d}\big[\,\big|\, (a'_{i1}\overline{D}_i)\ (\bar{a}'_{i1}\,y_i)\ (a'_{i3}\,y_i) \,\big| \\ + \big|\, (a'_{i1}\,y_i)\ (\bar{a}'_{i1}y_i)\ (a'_{i3}\overline{D}_i) \,\big|\, \big] + \bar{p}r = 0. \end{aligned}$$

The cubic curve C_3 corresponds to a cubic cut from F_3 by the quadric through C_3 and touching F_3 at P and $\overline{P}$. This quadric has the equation

$$L \equiv \big|\, (a'_{i1}\,y_i)\ \ (a'_{i1}\,y_i)\ \ A \,\big| = 0,$$

where A, B, C are the second-order determinants of a two column matrix whose columns are made up of the second-order determinants of the matrix $\|(\bar{a}'_{i1}D_i)\ \ (a'_{i3}D_i)\|$ and the matrix $\|(a'_{i1}\overline{D}_i)\ \ (a'_{i3}\overline{D}_i)\|$. The web of quadrics which touch F_3 at P and $\overline{P}$ cuts F_3 in a web of sextic curves of genus two which is invariant under J_B. The locus of an additional point of contact with F_3 of quadrics of the web is the invariant nonic of J_B. It lies on the cubic surface K which has the equation

$$p\overline{H} - \bar{p}H = 0.$$

If the a_{ij}, b_{ij}, and c_{ij} are polynomials of order n in λ, the surfaces H, $\overline{H}$, L, and K are of orders $48n+13$, $48n+13$, $24n+6$, and $36n+9$ in λ. When λ is replaced by x_4/x_3, then as in §2 we can write the table of characteristics of the involutorial space transformation $I_{120n+51}$ as follows:

$$\begin{aligned}
l &\sim F_{24n+12}: l^{24n+7+t} + (O,\overline{O})^{24n+8} + C^{2}_{12n+6},\\
(O,\overline{O}) &\sim F_{96n+38}: l^{96n+26+5t} + (O,\overline{O})^{96n+31} + C^{4}_{12n+6},\\
C_{12n+6} &\sim F_{264n+112}: l^{264n+76+12t} + (O,\overline{O})^{264n+88} + C^{13}_{12n+6},\\
S_1 &\sim S_{120n+51}: l^{120n+34+6t} + (O,\overline{O})^{120n+40} + C^{6}_{12n+6},\\
& K_{36n+18}: l^{36n+9+3t} + (O,\overline{O})^{36n+12} + C^{3}_{12n+6}.
\end{aligned}$$

The parasitic lines of the transformation are of three types. (a) There are the $24n+8$ trisecants of the C_{12n+6}. (b) The bisecants of C_{12n+6} through O or $\bar{O}$ are parasitic. In a plane λ through l there are fifteen bisecants through the six points of C_{12n+6} not on l, which determine fifteen points μ on l. Given a point μ on l there are $h'=36n+7$ bisecants of C_{12n+6} through it which determine h' planes λ. In the (λ, μ) correspondence there are $15+2h'$, ($2h'$, since $\mu^2=\lambda$), coincidences, or $72n+29$ positions of the points O, $\bar{O}$ such that bisecants of C_{12n+6} may be drawn through them. (c) In each of the $12n$ planes determined by l and the tangents to C_{12n+6} at its $12n$ intersections with l, the Bertini involution in the plane breaks down, and the line l is shed off. Hence there are $12n$ parasitic lines consecutive to l in these $12n$ planes. The total number of parasitic lines is $108n+37$.

To determine the number of parasitic conics and cubics we take the complete intersection of two $S_{120n+51}$ and an $S_{120n+51}$ with K_{36n+18}.

$$\begin{aligned}(120n + 51)^2 &= 120n + 51 + (120n + 34)^2 + 36 + 36(12n + 6)\\ &\quad +108n + 73 + 8\zeta + 27\eta,\\ (120n + 51)(36n + 18) &= 36n + 18 + (36n + 9)(120n + 34)\\ &\quad + 18 + 18(12n + 6) + 108n + 37 + 4\zeta + 9\eta.\end{aligned}$$

The solution of these equations is $\zeta=144n+47$ conics and $\eta=84n+27$ cubics. The fundamental curves of the second species of the involutorial transformation $I_{120n+51}$ consist of $108n+37$ lines, $144n+47$ conics, and $84n+27$ cubics.

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