A GENERALIZATION OF HARMONIC FUNCTIONALS*

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1. *Introduction.* In a recent paper W. V. D. Hodge† showed that most of the elementary properties of harmonic functions could be extended to harmonic functionals; on the other hand, we can extend the notion of harmonic functionals so that most of these elementary properties persist in this larger class. The present paper presents such an extension, and properties of the functionals it contains.

2. *Generalized Harmonic Forms.* Consider the $(p-1)$-form

$$\phi = A_{i_1 \cdots i_{p-1}} dx^{i_1} \cdots dx^{i_{p-1}}, \quad (i = 1, \cdots, n), \tag{1}$$

in which the elements concerned obey the usual laws, with the exception that the dx's obey the non-commutative law of multiplication

$$dx^i dx^j = - dx^j dx^i. \tag{2}$$

Without loss of generality we assume that the summation in (1) is taken over all i for which $i_1 < \cdots < i_{p-1}$. If the A's have second partial derivatives which are continuous, then the form ϕ is said to be regular. The properties of such forms have been discussed by Cartan.‡

If the coefficients $A_{i_1 \cdots i_{p-1}}$ are symbols, we shall speak of (1) as a symbolic form. In the present paper we shall be concerned only with symbolic linear forms:

$$\alpha = \alpha_i dx^i, \ \beta = \beta_i dx^i, \ \cdots .$$

The rules of combination for these are the same as for the forms of the type (1), except that the commutative law for the multiplication of a symbolic and a non-symbolic form does not hold.

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* Presented to the Society, February 23, 1935.

† W. V. D. Hodge, *A Dirichlet problem for harmonic functionals, with application to analytic varieties*, Proceedings of the London Mathematical Society, (2), vol. 36, part 4, pp. 257–303.

‡ E. Cartan, *Leçons sur les Invariants Intégraux*, 1922, Chapter 7.

In particular we note that if α_i, β_j are symbols and A is a function, the products $\alpha_i\beta_j=\beta_j\alpha_i$, $A\alpha_i$ are symbols; but $\alpha_i A$ is a function, since α_i has spent itself on the function A. Any form is said to be zero if all its coefficients are zero, and is indicated by writing $\phi=0$.

Let the multiplication of the form (1) by the symbolic linear form $\alpha=\alpha_i dx^i$ be indicated by ϕ_α,

$$\phi_\alpha = B_{i_1\cdots i_p}dx^{i_1}\cdots dx^{i_p},$$

$$B_{i_1\cdots i_p} = \sum_{k=1}^{p}(-1)^{k-1}\alpha_{i_k}A^{i_k}_{i_1\cdots i_p}, \quad (i_1<\cdots<i_p), \tag{3}$$

where

$$A^{i_k}_{i_1\cdots i_p} = A_{i_1\cdots i_{k-1}i_{k+1}\cdots i_p}.$$

Denoting the adjoint of ϕ_α by ϕ^α, we have

$$\phi^\alpha = \sum \pm B_{i_1\cdots i_p}dx^{i_{p+1}}\cdots dx^{i_n},$$

the $+$ or $-$ sign being chosen according as $i_1,\cdots,i_n$ is an even or odd derangement of $1,\cdots,n$. Finally, let ϕ^α be multiplied by the symbolic form $\beta=\beta_i dx^i$, and let the result be indicated by

$$\Delta_\beta{}^\alpha\phi. \tag{4}$$

If γ is a cycle of $(p-1)$ dimensions, we shall define the integral $\int_\gamma\phi$ as an $\alpha\beta$-functional of γ if the form (4) is zero. Also in such a case, we shall speak of ϕ as an $\alpha\beta$-form.* We observe that if $\alpha_i=\beta_i=\partial/\partial x^i$, then ϕ_α is known in the literature as the covariant derivative of ϕ; and that if $\Delta_\beta{}^\alpha\phi=0$, and ϕ_α is regular, ϕ is a harmonic form and $\int_\gamma\phi$ a harmonic functional of γ.

3. *Properties of the $\alpha\beta$-forms.* We now give a series of theorems which are generalizations of the theorems of harmonic forms.

THEOREM 1. *The $B_{i_1\cdots i_p}$ defined by* (3) *satisfy the relations*

$$\sum_{k=1}^{p+1}(-1)^{k-1}\alpha_{i_k}B^{i_k}_{i_1\cdots i_{p+1}} = 0. \tag{5}$$

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* A more general functional could be obtained by taking α and β as any symbolic forms.

From the properties assigned to our symbolic forms, we see that the associative law holds, so that

$$\phi_{\alpha\alpha} = \alpha\alpha\phi = 0\cdot\phi = 0,$$

where $\alpha\alpha=0$ by virtue of (2); hence the theorem is immediate since the left side of (5) is a coefficient of $\phi_{\alpha\alpha}$.

THEOREM 2. *If $p=1$ and ϕ is an $\alpha\beta$-form, then*

$$\beta_1\alpha_1\phi + \cdots + \beta_n\alpha_n\phi = 0. \tag{6}$$

The proof is easily supplied. We remark that any function satisfying the relation (6) will be called an $\alpha\beta$-function.

THEOREM 3. *If ϕ is an $\alpha\beta$-form, then the coefficients of ϕ_α are $\alpha\beta$-functions.*

Using the definition of ϕ^α, we have

$$\Delta_\beta{}^\alpha\phi = \sum \pm\left(\sum_{j=1}^{p} \beta_{i_j}B_{i_1\cdots i_p}dx^{i_j}\right)dx^{i_{p+1}}\cdots dx^{i_n} = 0, \tag{7}$$

and wish to show that

$$\alpha_j\beta_jB_{i_1\cdots i_p} = 0 \tag{8}$$

for each set $i_1<\cdots<i_p$.

We shall make use of the notation $B^{i_r}_{i_1\cdots i_p,i_{p+v}}$ to mean that B in which i_r is missing from $i_1<\cdots<i_p$ and that i_{p+v} is to be put into its natural order in $i_1<\cdots<i_p$, a minus or a plus sign being taken according as i_{p+v} is taken over an odd or over an even number of i's. Note in (7), when finding the coefficient of $dx^{i_r}dx^{i_{p+1}}\cdots dx^{i_n}$, where $1\leqq r\leqq p$, that besides the term $\beta_{i_r}B_{i_1\cdots i_p}$, two types of terms can appear, those for which $i_{p+v}<i_r$ and those for which $i_{p+v}>i_r$. Taking this into account, we can write, except perhaps for a sign, this coefficient in the form

$$\beta_{i_r}B_{i_1\cdots i_p} + (-1)^{p-r}\sum_{v=1}^{n-p}\beta_{i_{p+v}}B^{i_r}_{i_1\cdots i_p,i_{p+v}}.$$

For an $\alpha\beta$-form the above symbol is zero, hence multiplying by α_{i_r} and summing with respect to r from 1 to p, we have

$$(9)\quad (-1)^p \sum_{r=1}^{p} \alpha_{i_r}\beta_{i_r} B_{i_1\cdots i_p} = \sum_{v=1}^{n-p} \beta_{i_{p+v}} \sum_{r=1}^{p} (-1)^{r-1} \alpha_{i_r} B^{i_r}_{i_1\cdots i_p,\, i_{p+v}},$$

where we have reversed the order of summation on the right hand side. If now $i_{k_v} < i_{p+v} < i_{k_v+1}$, where $k_v < p$, we can adjust the notation, so that Theorem 1 reads

$$\sum_{r=1}^{k_v} (-1)^{p+r-1} \alpha_{i_r} B^{i_r}_{i_1\cdots i_p,\, i_{p+v}} + \alpha_{i_{p+v}} B_{i_1\cdots i_p}$$
$$+ \sum_{r=k_v+1}^{p} (-1)^{p+r-1} \alpha_{i_r} B^{i_r}_{i_1\cdots i_p,\, i_{p+v}} = 0.$$

Using this in (9), we get

$$\sum_{r=1}^{p} \alpha_{i_r}\beta_{i_r} B_{i_1\cdots i_p} = - \sum_{v=1}^{n-p} \beta_{i_{p+v}} \alpha_{i_{p+v}} B_{i_1\cdots i_p},$$

which leads immediately to the theorem.

4. *Forms in Two Sets of Differentials.* Turn now to the special case of $A_{i_1\cdots i_{p-1}} dx^{i_1} \cdots dx^{i_{p-1}} d\xi^{i_1} \cdots d\xi^{i_{p-1}}$, namely,*

$$U = A dx^{i_1} \cdots dx^{i_{p-1}} d\xi^{i_1} \cdots d\xi^{i_{p-1}}.$$

Let $\eta = \eta_i d\xi^i$ be a linear symbolic form and multiply U by it. Then

$$U_\eta = \left(\sum_{j=1}^{p} (-1)^{j-1} \eta_{i_j} A dx^{i_1} \cdots dx^{i_{j-1}} dx^{i_{j+1}} \cdots dx^{i_p} \right) d\xi^{i_1} \cdots d\xi^{i_p}.$$

The following theorem will be found useful.

THEOREM 4. *If $\beta_i A = \eta_i A$ (or if $\beta_i A = -\eta_i A$) and if A is an $\alpha\beta$-function, then U_η is an $\alpha\beta$-form.*

We may write $U_{\eta\alpha}$ in the form

$$U_{\eta\alpha} = \left(\sum_{j=1}^{p} \alpha_{i_j}\eta_{i_j} A dx^{i_1} \cdots dx^{i_p} + \sum_{k=p+1}^{n} \sum_{j=1}^{p} (-1)^{j-1} \right.$$
$$\left. \cdot\, \alpha_{i_k}\eta_{i_j} A dx^{i_k} dx^{i_1} \cdots dx^{i_{j-1}} dx^{i_{j+1}} \cdots dx^{i_p} \right) d\xi^{i_1} \cdots d\xi^{i_p}.$$

* The dx and $d\xi$ satisfy the law (2) separately, but in the present paper we assume that dx is commutative with $d\xi$. With this assumption U can be looked on as a symbolic form with $A d\xi^{i_1} \cdots d\xi^{i_{p-1}}$ as coefficients.

Forming the adjoint of the above with respect to the dx's, we have

$$(U_\eta)^\alpha = \sum \pm \left(\sum_{j=1}^{p} \alpha_{i_j}\eta_{i_j} A dx^{i_{p+1}} \cdots dx^{i_n} + \sum_{k=p+1}^{n} \sum_{j=1}^{p} (-1)^{k+p} \right.$$

$$\left. \cdot\, \alpha_{i_k}\eta_{i_j} A dx^{i_j} dx^{i_{p+1}} \cdots dx^{i_{k-1}} dx^{i_{k+1}} \cdots dx^{i_n} \right) d\xi^{i_1} \cdots d\xi^{i_p}.$$

Multiplying by β gives

$$\Delta_\beta{}^\alpha U_\eta = \sum \pm \left(\sum_{v=1}^{p} \beta_{i_v} \left[\sum_{j=1}^{p} \alpha_{i_j}\eta_{i_j} A \right] dx^{i_v} dx^{i_{p+1}} \cdots dx^{i_n} \right.$$

$$+ \sum_{k=p+1}^{n} \sum_{j=1}^{p} \beta_{i_k}\alpha_{i_k}\eta_{i_j} A dx^{i_j} dx^{i_{p+1}} \cdots dx^{i_n} + \sum_{v=1}^{p} \sum_{j=1}^{p} \sum_{k=p+1}^{n} (-1)^{k+p}$$

$$\left. \beta_{i_v}\eta_{i_j}\alpha_{i_k} A dx^{i_v} dx^{i_j} dx^{i_{p+1}} \cdots dx^{i_{k-1}} dx^{i_{k+1}} \cdots dx^{i_n} \right) d\xi^{i_1} \cdots d\xi^{i_p}.$$

If now $\beta_i A = \eta_i A$, we find that $\Delta_\beta{}^\alpha U_\eta$ is equal to

$$\sum \pm \left(\sum_{v=1}^{p} \eta_{i_v} \left[\sum_{j=1}^{n} \beta_{i_j}\alpha_{i_j} A \right] dx^{i_v} dx^{i_{p+1}} \cdots dx^{i_n} \right) d\xi^{i_1} \cdots d\xi^{i_p}. \tag{10}$$

The expression (10) has the opposite sign if $\beta_i A = -\eta_i A$. Thus if A satisfies the equation

$$\beta_1\alpha_1 A + \cdots + \beta_n\alpha_n A = 0,$$

U_η is an $\alpha\beta$-form.

If U^η is the adjoint of U_η with respect to the $d\xi$'s, we readily see from (10) that we have the following corollary.

COROLLARY. *If the conditions of Theorem* 4 *hold, then* U^η *is an* $\alpha\beta$*-form.*

4. *Special Cases.* In what follows we refer to a real euclidean space of n-dimensions, with $(x^1, \cdots, x^n)$ as a system of rectangular cartesian axes. The case treated by Hodge is obtained by specializing α and β to be

$$\alpha = \beta = \frac{\partial}{\partial x^i} dx^i. \tag{11}$$

With this choice of α and β, Theorem 2 takes the form of Laplace's equation

$$\left(\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2}\right)\phi = 0, \qquad (x^i = x_i),$$

hence the name harmonic forms. With $\eta=(\partial/\partial\xi^i)d\xi^i$, the A of Theorem 4 can be taken

$$A = \frac{1}{r^{n-2}}, \text{ when } n > 2,$$
$$= \log r, \text{ when } n = 2,$$

where $r^2=\Sigma(x^i-\xi^i)^2$. If we take

$$\alpha = \frac{\partial}{\partial x^i}\, dx^i,$$

$$\beta = \frac{\partial}{\partial x^1}\, dx^1 + \cdots + \frac{\partial}{\partial x^{n-1}}\, dx^{n-1} - dx^n, \tag{12}$$

$$\eta = \frac{\partial}{\partial \xi^1}\, d\xi^1 + \cdots + \frac{\partial}{\partial \xi^{n-1}}\, d\xi^{n-1} - d\xi^n,$$

then Theorem 2 takes the form of the parabolic equation

$$\left(\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_{n-1}^2}-\frac{\partial}{\partial x_n}\right)\phi = 0.$$

Hence if α and β are defined by (12), we shall speak of an $\alpha\beta$-form as a parabolic form, and $\int_\gamma \phi$ as a parabolic functional. The A of Theorem 4 can be taken as

$$A = \frac{1}{(x_n - \xi_n)^{(n-1)/2}}\, e^{-[(x_1-\xi_1)^2+\cdots+(x_{n-1}-\xi_{n-1})^2]/[4(x_n-\xi_n)]}.$$

If α and β are defined by (11), with the exception that $\alpha_n=-\partial/\partial x^n$, then (6) of Theorem 2 takes the form of the hyperbolic equation, and we can refer to such $\alpha\beta$-forms as hyperbolic forms.

5. *Green's Theorem for Parabolic Functionals.* If ϕ and ψ are regular $(p-1)$-forms, Hodge has proved the following theorem of Green useful in the case of harmonic functionals:

THEOREM. *If α and β are given by* (11) *and D is any domain in our space bounded by the contour γ, we have*

$$\int_D (\Delta_\beta{}^\alpha \phi)\cdot\psi - \int_D (\Delta_\beta{}^\alpha \psi)\cdot\phi = \int_\gamma \phi^\alpha\cdot\psi - \int_\gamma \psi^\alpha\cdot\phi.*$$

We shall now prove a corresponding theorem applicable to the case of parabolic functionals. Let α and β be defined by (12) and let

$$\bar\beta = \bar\beta_i dx^i = \frac{\partial}{\partial x^1}dx^1 + \cdots + \frac{\partial}{\partial x^{n-1}}dx^{n-1} + dx^n;$$

then our final theorem reads:

THEOREM 5. *If D is a region in our space bounded by the contour γ, and ϕ and ψ are regular $(p-1)$-forms, then*

$$\int_D (\Delta_\beta{}^\alpha \psi)\cdot\phi - \int_D (\Delta_\alpha{}^{\bar\beta} \phi)\cdot\psi = \int_\gamma [\psi^\alpha\cdot\phi]_n - \int_\gamma \phi^{\bar\beta}\cdot\psi,$$

where $[\psi^\alpha\cdot\phi]_n$ means that terms of $\psi^\alpha\cdot\phi$ not containing a dx^n are dropped.

Let

$$\psi = C_{i_1\cdots i_{p-1}} dx^{i_1}\cdots dx^{i_{p-1}},$$
$$\phi = A_{i_1\cdots i_{p-1}} dx^{i_1}\cdots dx^{i_{p-1}}.$$

Making use of (3) and (7), we have

$$(\Delta_\beta{}^\alpha\psi)\cdot\phi = (-1)^{n(p-1)} \cdot\sum \pm \left(\sum_{j=1}^p\sum_{k=1}^p (-1)^{j+k} A^{i_j}_{i_1\cdots i_p}\beta_{i_j}\alpha_{i_k} C^{i_k}_{i_1\cdots i_p}\right) dx^{i_1}\cdots dx^{i_n}.$$

Forming $(\Delta_\alpha{}^{\bar\beta}\phi)\cdot\psi$ and subtracting, we obtain

$$(13)\qquad \begin{aligned}(\Delta_\beta{}^\alpha\psi)\phi - (\Delta_\alpha{}^{\bar\beta}\phi)\psi = (-1)^{n(p-1)}\sum \pm \Bigg(\sum_{j=1}^p\sum_{k=1}^p (-1)^{j+k} \\ \cdot\left[A^{i_j}_{i_1\cdots i_p}\beta_{i_j}\alpha_{i_k}C^{i_k}_{i_1\cdots i_p} - C^{i_k}_{i_1\cdots i_p}\alpha_{i_k}\bar\beta_{i_j}A^{i_j}_{i_1\cdots i_p}\right]\Bigg) dx^{i_1}\cdots dx^{i_n}.\end{aligned}$$

* For proof see W. V. D. Hodge, loc. cit., p. 266.

Two distinct cases arise:

Case 1. $\beta_{i_j} \neq \beta_n$,

$$A^{i_j}_{i_1 \cdots i_p} \beta_{i_j} \alpha_{i_k} C^{i_k}_{i_1 \cdots i_p} - C^{i_k}_{i_1 \cdots i_p} \alpha_{i_k} \bar{\beta}_{i_j} A^{i_j}_{i_1 \cdots i_p}$$
$$= \beta_{i_j}(A^{i_j}_{i_1 \cdots i_p} \alpha_{i_k} C^{i_k}_{i_1 \cdots i_p}) - \alpha_{i_k}(C^{i_k}_{i_1 \cdots i_p} \bar{\beta}_{i_j} A^{i_j}_{i_1 \cdots i_p}).$$

Case 2. $\beta_{i_j} = \beta_n$,

$$A^{i_j}_{i_1 \cdots i_p} \beta_{i_j} \alpha_{i_k} C^{i_k}_{i_1 \cdots i_p} - C^{i_k}_{i_1 \cdots i_p} \alpha_{i_k} \bar{\beta}_{i_j} A^{i_j}_{i_1 \cdots i_p}$$
$$= - \alpha_{i_k}(C^{i_k}_{i_1 \cdots i_p} \bar{\beta}_{i_j} A^{i_j}_{i_1 \cdots i_p}).$$

A term from Case 1 multiplied by $dx^{i_1} \cdots dx^{i_n}$ and integrated leaves terms of the following type to be integrated over γ:

$$(-1)^{j-1} A^{i_j}_{i_1 \cdots i_p} \alpha_{i_k} C^{i_k}_{i_1 \cdots i_p} dx^{i_1} \cdots dx^{i_{j-1}} dx^{i_{j+1}} \cdots dx^{i_n}$$
$$- (-1)^{k-1} C^{i_k}_{i_1 \cdots i_p} \bar{\beta}_{i_j} A^{i_j}_{i_1 \cdots i_p} dx^{i_1} \cdots dx^{i_{k-1}} dx^{i_{k+1}} \cdots dx^{i_n};$$

whereas from Case 2, we get on integration but the one term

$$- (-1)^{k-1} C^{i_k}_{i_1 \cdots i_p} \bar{\beta}_{i_j} A^{i_j}_{i_1 \cdots i_p} dx^{i_1} \cdots dx^{i_{k-1}} dx^{i_{k+1}} \cdots dx^{i_n}.$$

Thus if we integrate (13) over D and make use of the above results, we see that the theorem follows.

It is interesting to note that for $n=2$, $p=1$, Theorem 5 gives

$$\int_D \left\{ \left[\frac{\partial^2 \psi}{\partial x_1^2} - \frac{\partial \psi}{\partial x_2} \right] \phi - \left[\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial \phi}{\partial x_2} \right] \psi \right\} dx_1 dx_2$$
$$= \int_\gamma \phi \psi dx_1 + \left(\frac{\partial \psi}{\partial x_1} \phi - \frac{\partial \phi}{\partial x_1} \psi \right) dx_2.$$

This result is well known in the literature.*

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* Boundary value problems for the new functionals will be presented in a later article.