THE APPROXIMATE SOLUTION OF INTEGRAL EQUATIONS*

BY E. N. OBERG

1. *Introduction.* Consider the Fredholm integral equation

$$L(u) \equiv u(x) - \int_a^b k(x, t)u(t)dt = f(x). \tag{1}$$

Let it be assumed that the given function $f(x)$ is continuous in the interval $a \leqq x \leqq b$, and that the kernel $k(x, t)$ is continuous in the square $a \leqq x \leqq b$, $a \leqq t \leqq b$. Let us assume also that the equation $L(u) = 0$ has no non-trivial solution. Then a unique continuous solution exists for the unknown $u(x)$ of (1) of the form†

$$u(x) = f(x) + \int_a^b H(x, t)f(t)dt, \tag{2}$$

in which the resolvent kernel $H(x, t)$ is a well-determined continuous function in the square $a \leqq x \leqq b$, $a \leqq t \leqq b$.

Let

$$P_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

be an arbitrary polynomial of degree n. Then a problem in minima is to determine the coefficients of this polynomial so that the integral

$$\int_a^b |f(x) - L(P_n)|^m dx \tag{3}$$

shall be a minimum, where m is any positive real number.

The purpose of this paper is to examine the existence and uniqueness of such a polynomial and its convergence towards

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† See, for example, Courant-Hilbert, *Methoden der mathematischen Physik*, 2d ed., 1931, vol. 1, pp. 121–124. Less restrictive hypotheses can be placed on $k(x, t)$ (see, for example, E. W. Hobson, *On linear integral equations*, Proceedings of the London Mathematical Society, vol. 13 (1913–1914), pp. 307–340) but for the present discussion the hypothesis mentioned above may be regarded as sufficiently illustrative.

$u(x)$ as n becomes infinite. Various investigations of a similar nature have been conducted by Krawtchouk,* Enskog,† and Picone,‡ but by methods different from ours and in every case for $m > 1$. Picone has considered the same problem as ours but only for the special case $m = 2$, and by a method which does not appear capable of extension to other values of m. Further mention should also be made of a paper by McEwen§ on linear differential equations to which the following paper is in many aspects an analog.

2. *The Existence and Uniqueness of an Approximating Polynomial.* The questions of existence and uniqueness of a polynomial minimizing (3) can be disposed of readily by application of theorems which are already well known on least mth power approximation.¶ The problem can be looked upon as that of approximating the given function $f(x)$ by a linear combination of the $n+1$ continuous functions $L(1)$, $L(x)$, $L(x^2), \cdots, L(x^n)$. These functions are linearly independent in the interval $a \leqq x \leqq b$, since to assume otherwise would lead to the conclusion that a polynomial $p_n(x)$ exists, not identically equal to zero, which satisfies the homogeneous equation $L(p_n) = 0$ in contradiction with the hypothesis placed on $L(u)$. It follows from the general theorems to which reference has been made that for $m > 0$ a minimizing polynomial exists, and that for $m > 1$ this polynomial is unique.

3. *Convergence of the Approximating Polynomial for $m \geqq 1$.* Let $P_n(x)$ be the polynomial which minimizes (3), and $u(x)$ the unique continuous solution of (1). Let $q_n(x)$ be any nth degree polynomial, and ϵ_n a corresponding upper bound for the absolute value of $r_n(x) = u(x) - q_n(x)$. Let γ_n be the minimum of (3):

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* See M. Krawtchouk, *Sur la résolution approchée des équations intégrales linéaires*, Comptes Rendus (Paris), vol. 188 (1929), pp. 978–980.

† See D. Enskog, *Eine allgemeine Methode zur Auflösung von linearen Integralgleichungen*, Mathematische Zeitschrift, vol. 24 (1926), pp. 670–682.

‡ M. Picone, *Sul metodo delle minime potenze ponderate e sul metodo di Ritz*, Rendiconti del Circolo Matematico di Palermo, vol. 52 (1928), pp. 225–254.

§ See W. H. McEwen, *Problems of closest approximation connected with the solution of linear differential equations*, Transactions of this Society, vol. 33 (1931), pp. 979–997, and also this Bulletin, vol. 38 (1933), pp. 887–894.

¶ For a proof, see, for example, D. Jackson, *On functions of closest approximation*, Transactions of this Society, vol. 22 (1921), pp. 117–128.

$$
(4)\qquad
\begin{aligned}
\gamma_n &= \int_a^b \left| f(x) - L(P_n) \right|^m dx = \int_a^b \left| L(u) - L(P_n) \right|^m dx \\
&= \int_a^b \left| L(u - P_n) \right|^m dx.
\end{aligned}
$$

Since $q_n(x)$ is also a polynomial of the nth degree, it follows from the minimizing property of γ_n that

$$
(5)\qquad
\begin{aligned}
\gamma_n &= \int_a^b \left| L(u - P_n) \right|^m dx \\
&\leqq \int_a^b \left| L(u - q_n) \right|^m dx = \int_a^b \left| L(r_n) \right|^m dx.
\end{aligned}
$$

But

$$
L(r_n) = r_n(x) - \int_a^b k(x, t) r_n(t) dt,
$$

whence if $|k(x, t)| \leqq M$,

$$
(6)\qquad
\begin{aligned}
\left| L(r_n) \right| &\leqq \left| r_n(x) \right| + \int_a^b \left| k(x, t) \right| \left| r_n(t) \right| dt \\
&\leqq \left| 1 + M(b - a) \right| \epsilon_n.
\end{aligned}
$$

Hence if (6) is substituted in (5), it follows that

$$
(7)\qquad \gamma_n \leqq (N\epsilon_n)^m,
$$

where N is a positive constant not depending on n or ϵ_n.

If the difference $P_n(x) - q_n(x)$ is denoted by $\pi_n(x)$, then $r_n - \pi_n$ is the same as $u - P_n$. Let $\phi_n(x) = r_n(x) - \pi_n(x)$, and let $z(x)$ represent the continuous function $L(\phi_n)$:

$$
(8)\qquad z(x) = \phi_n(x) - \int_a^b k(x, t) \phi_n(t) dt.
$$

From (2),

$$
\phi_n(x) = z(x) + \int_a^b H(x, t) z(t) dt,
$$

and if $|H(x, t)| \leqq G$ throughout the square $a \leqq x \leqq b$, $a \leqq t \leqq b$, then

$$(9) \qquad | \phi_n(x) - z(x) | \leqq G \int_a^b | z(t) | \, dt.$$

The integrand on the right is non-negative and continuous in the closed interval (a, b), so that a form of Hölder's inequality* can be applied to (9) with the result that

$$| \phi_n(x) - z(x) | \leqq G(b - a)^{(m-1)/m} \left[\int_a^b | z(t) |^m dt \right]^{1/m}.$$

The value of the integral in the brackets is γ_n, and by virtue of (7),

$$| \phi_n(x) - z(x) | \leqq G(b - a)^{(m-1)/m} \gamma_n^{1/m} \leqq \mu\epsilon_n.$$

For convenience in the use of the notation later, it will be understood that μ represents the greater of the two quantities 1 and $G(b-a)^{(m-1)/m}N$. But from (8),

$$\left| \int_a^b k(x, t)\phi_n(t)dt \right| = | \phi_n(x) - z(x) |,$$

whence

$$(10) \qquad \left| \int_a^b k(x, t)\phi_n(t)dt \right| \leqq \mu\epsilon_n.$$

Let the maximum of $|\pi_n(x)|$ for $a \leqq x \leqq b$ be $\mu\sigma_n$, and x_0 a point in the interval where $|\pi_n(x_0)| = \mu\sigma_n$. Then from Markoff's theorem,†

$$| \pi_n'(x) | \leqq 2\mu n^2\sigma_n/(b - a)$$

for all values of x in $a \leqq x \leqq b$. For $|x - x_0| \leqq (b-a)/4n^2$, by the mean value theorem,

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* We are applying the inequality

$$\int_a^b F(x)dx \leqq (b-a)^{(P-1)/P}\left[\int_a^b [F(x)]^P dx\right]^{1/P},$$

where $F(x)$ is assumed $\geqq 0$, and $P \geqq 1$. It is because of the fact that Hölder's inequality applies for $P \geqq 1$ only, that we must break up the convergence proof into two parts, one for $m \geqq 1$, and the other for $m < 1$, which must be approached by a different method.

† See, for example, D. Jackson, Transactions of this Society, vol. 22 (1921), p. 163.

$$\left| \pi_n(x) - \pi_n(x_0) \right| \leqq \mu\sigma_n/2,$$

from which it follows that

$$\left| \pi_n(x) \right| \geqq \mu\sigma_n/2.$$

Let it be assumed for the time being that ϵ_n is less than or at most equal to $\sigma_n/4$ (the contrary case which leads directly to the desired conclusion will be discussed later). Then

$$\begin{aligned} \left| \phi_n(x) \right| &= \left| \pi_n(x) - r_n(x) \right| \geqq \left| \pi_n(x) \right| - \left| r_n(x) \right| \\ &\geqq \frac{\mu}{2}\sigma_n - \frac{1}{4}\sigma_n \geqq \frac{\mu}{2}\sigma_n - \frac{\mu}{4}\sigma_n = \frac{\mu}{4}\sigma_n. \end{aligned}$$

With (10) this gives

$$\begin{aligned} \left| L(\phi_n) \right| &\geqq \left| \phi_n(x) \right| - \left| \int_a^b k(x, t)\phi_n(t)dt \right| \\ &\geqq \frac{\mu}{4}\sigma_n - \mu\epsilon_n = \mu\left(\frac{\sigma_n}{4} - \epsilon_n\right), \end{aligned}$$

or, since at least one-half of the interval $|x-x_0| \leqq (b-a)/(4n^2)$ is contained in (a, b) and since $\phi_n = r_n - \pi_n = u - P_n$,

$$\gamma_n \geqq \frac{b-a}{4n^2}\left| \mu\left(\frac{\sigma_n}{4} - \epsilon_n\right)\right|^m.$$

Under the assumption that $\epsilon_n \leqq \sigma_n/4$ it is found that

$$\sigma_n \leqq \frac{4 \cdot 4^{1/m}}{(b-a)^{1/m}} n^{2/m} \gamma_n^{1/m} + 4\epsilon_n. \tag{11}$$

On the other hand, if the assumption is made that ϵ_n is greater than $\sigma_n/4$, then $\sigma_n < 4\epsilon_n$, so that (11) is generally true.

Furthermore, since $|\pi_n(x)| \leqq \mu\sigma_n$ and $|r_n(x)| \leqq \epsilon_n$, it follows that

$$\left| \phi_n(x) \right| = \left| r_n(x) - \pi_n(x) \right| \leqq \epsilon_n + \mu\sigma_n.$$

But $r_n - \pi_n$ is identical with $u - P_n$; hence

$$\left| u(x) - P_n(x) \right| \leqq \frac{4 \cdot 4^{1/m}}{(b-a)^{1/m}} n^{2/m} \gamma_n^{1/m} + 4\mu\epsilon_n + \epsilon_n.$$

It is now apparent that, for n sufficiently large, since $\gamma_n \leqq (N\epsilon_n)^m$,

$$| u(x) - P_n(x) | \leqq S n^{2/m} \epsilon_n,$$

where S is a positive constant.

The previous discussion can be summed up as follows.

THEOREM 1. *If $u(x)$ is a solution of the integral equation*

$$L(u) \equiv u(x) - \int_a^b k(x, t)u(t)dt = f(x),$$

under the hypothesis on $L(u)$ already stated, and if $P_n(x)$ is the approximating polynomial to $u(x)$ determined by the least mth power method, then a sufficient condition for the convergence of $P_n(x)$ towards $u(x)$ is that it be possible to choose polynomials $q_n(x)$ for every value of n so that

$$\lim_{n\to\infty} n^{2/m}\epsilon_n = 0,$$

where ϵ_n is an upper bound for $|u(x) - q_n(x)|$ in the interval (a, b).

The last condition can be interpreted in terms of continuity of $u(x)$ and its derivatives, and these in turn will be guaranteed by imposition of suitable hypotheses on the given functions $f(x)$ and $k(x, t)$. It appears from the representation

$$u(x) = f(x) + \int_a^b k(x, t)u(t)dt$$

that if $f(x)$ has a modulus of continuity not exceeding $\omega(\delta)$, and if $k(x, t)$ as a function of x has, uniformly with respect to t, a modulus of continuity not exceeding a constant multiple of $\omega(\delta)$, then $u(x)$ likewise has a modulus of continuity not exceeding a constant multiple of $\omega(\delta)$. For the case $m > 2$, the condition that it be possible to make $n^{2/m}\epsilon_n$ approach zero will be satisfied, according to known theorems on approximation,* if $\lim_{\delta\to 0} \omega(\delta)/\delta^{2/m} = 0$. In a similar way it is seen from the representation

$$u^{(p)}(x) = f^{(p)}(x) + \int_a^b \frac{\partial^p}{\partial x^p} k(x, t)u(t)dt$$

* D. Jackson, *The Theory of Approximation*, American Mathematical Society Colloquium Publications, vol. XI, 1930, pp. 13–18.

for the pth derivative of $u(x)$ that, when $1<m\leqq 2$, sufficient conditions for convergence can be formulated in terms of properties of continuity of $f(x)$, $k(x, t)$, and $k_x(x, t)$, and that for $m=1$ there are corresponding conditions involving second derivatives.

4. *Convergence of the Approximating Polynomial for* $m<1$. In order to proceed with the discussion of convergence of the approximating polynomial $P_n(x)$ for $m<1$, there is occasion to add to the hypothesis already assumed for equation (1) the condition that $k_x(x, t)$, the derivative of $k(x, t)$ with respect to x, be a continuous function of the two variables in the closed region $a\leqq x\leqq b$, $a\leqq t\leqq b$.

Before presentation of the actual convergence theorem, the following auxiliary theorem will be established.

If $p_n(x)$ is an arbitrary polynomial of the nth degree, and $L(u)$ the expression defined in (1), *then, if η is the maximum of $|L(p_n)|$ on $a\leqq x\leqq b$,*

$$|p_n(x)| \leqq A\eta,$$

and

$$\left|\frac{d}{dx}L[p_n(x)]\right| \leqq Bn^2\eta$$

for all values of x in (a, b), where A and B are positive constants depending neither on n nor on the coefficients in $p_n(x)$.

Let

$$R(x) = L(p_n) = p_n(x) - \int_a^b k(x, t)p_n(t)dt.$$

Since

$$p_n(x) = R(x) + \int_a^b H(x, t)R(t)dt,$$

it follows that

$$|p_n(x)| \leqq |R(x)| + \int_a^b |H(x, t)||R(t)|dt$$

$$\leqq \left[1 + \int_a^b Gdt\right]\eta = A\eta,$$

where $A=1+G(b-a)$. Moreover, as it was assumed that $k_x(x, t)$ is continuous in $a\leqq x\leqq b$, $a\leqq t\leqq b$,

$$\frac{d}{dx}[L(p_n)] = p_n'(x) - \int_a^b k_x(x, t)p_n(t)dt,$$

whence

$$\left|\frac{d}{dx}[L(p_n)]\right| \leqq |p_n'(x)| + \int_a^b |k_x(x, t)||p_n(t)|dt.$$

The function $p_n(x)$ is a polynomial of the nth degree; hence, by Markoff's theorem, $|p_n'(x)|\leqq N'n^2\eta$, where N' is a constant independent of η and the polynomial $p_n(x)$. It follows, therefore, if $|k_x(x, t)|\leqq M'$ in (a, b), that

$$\left|\frac{d}{dx}[L(p_n)]\right| \leqq N'n^2\eta + M'A(b-a)\eta.$$

Consequently,

$$\left|\frac{d}{dx}[L(p_n)]\right| \leqq Bn^2\eta,$$

where B is a constant.

To proceed with the convergence theorem, let x_0 be a point in $a\leqq x\leqq b$ at which $|L(\pi_n)|$ attains its maximum, where $\pi_n(x)$ has the same meaning as in the preceding section, and let this maximum be denoted by η. Then if x is interior to the interval $|x-x_0|\leqq 1/(2n^2B)$, or the part of this interval which is contained in (a, b), in case x_0 is distant from a or b by less than the amount indicated, and, if the mean value theorem is applied to $L(\pi_n)$ together with the conclusions of the above auxiliary theorem, it is seen that

$$|L[\pi_n(x)] - L[\pi_n(x_0)]| \leqq \eta/2,$$

whence $|L(\pi_n)|\geqq\eta/2$. But

$$|L(r_n)| \leqq |r_n(x)| + \int_a^b |k(x,t)||r_n(t)\ dt \leqq |1 + M(b-a)|\epsilon_n.$$

So if for the time being it be assumed that $|1+M(b-a)|\epsilon_n\leqq\eta/4$,

$$| L(\pi_n) - L(r_n) | \geqq | L(\pi_n) | - | L(r_n) | \geqq \eta/4$$

throughout the interval specified. Hence

$$\gamma_n = \int_a^b | L(\pi_n) - L(r_n) |^m dx \geqq \frac{1}{2n^2 B}\left(\frac{\eta}{4}\right)^m,$$

from which it follows that

$$\eta \leqq 4(2Bn^2\gamma_n)^{1/m}.$$

If, on the other hand, we assume that $[1+M(b-a)]\epsilon_n \geqq \eta/4$, then

$$\eta \leqq 4[1 + M(b - a)]\epsilon_n.$$

It is therefore evident that in all cases

$$\eta \leqq 4(2Bn^2\gamma_n)^{1/m} + 4[1 + M(b - a)]\epsilon_n.$$

Since by the auxiliary theorem $|\pi_n(x)| \leqq A\eta$, while $|r_n(x)| \leqq \epsilon_n$,

$$| r_n(x) - \pi_n(x) | \leqq | r_n(x) | + | \pi_n(x) | \leqq 4A(2Bn^2\gamma_n)^{1/m} + 4A[1 + M(b - a)]\epsilon_n + \epsilon_n.$$

Furthermore, since $r_n - \pi_n$ is the same as $u - P_n$ and $\gamma_n \leqq (N\epsilon_n)^m$,

$$| u(x) - P_n(x) | \leqq A' n^{2/m}\epsilon_n$$

for all values of x in the interval (a, b), where A' is a positive constant not depending on n or on ϵ_n. The following theorem can therefore be stated for the case $m<1$.

THEOREM 2. *If in addition to the hypothesis of Theorem* 1, *the assumption is made that $k_x(x, t)$ is continuous in the square $a \leqq x \leqq b$, $a \leqq t \leqq b$, then the conclusion of Theorem* 1 *is valid for $m<1$.*

The conditions to be imposed on $f(x)$ and $k(x, t)$, in order that it may be possible to make $n^{2/m}\epsilon_n$ approach zero, are similar to those which were mentioned after Theorem 1 for the case $1<m\leqq 2$, with suitably modified specifications involving second derivatives or derivatives of higher order.

THE UNIVERSITY OF MINNESOTA