IRREDUCIBILITY OF POLYNOMIALS OF DEGREE n WHICH ASSUME THE SAME VALUE n TIMES*

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1. *Introduction*. A polynomial $F(x)$ of degree n, with integral coefficients, which assumes the same value k for n distinct integral values of x has the form

$$F(x) = a_0(x - a_1)(x - a_2) \cdots (x - a_n) + k, \qquad (a_0 \neq 0),$$

where the a's denote integers, and $a_1, a_2, \cdots, a_n$ are distinct. The irreducibility of polynomials of this type in the field of rational numbers has been discussed by several writers for the particular cases† $|k| = 1$, $|k| =$ prime.

The present paper is concerned with the irreducibility of $F(x)$ for the case in which k is any integer $\neq 0$. It is obvious that even when the a's are fixed, an infinitude of choices of k exists for which $F(x)$ is reducible. What is not obvious is that when k and n are fixed, *only a finite number of non-equivalent reducible polynomials of the form* $F(x)$ *exist*. Two polynomials $F(x)$ and $G(x)$, with integral coefficients, are regarded as *equivalent* if an integer h exists such that $F(x) = \pm G(\pm x + h)$. Moreover, if only k is fixed, but n is sufficiently large, *every* polynomial of the form of $F(x)$ is irreducible.

2. *Isolation of the Roots of* $f(x)$. The polynomial $F(x)$ of §1 is evidently equivalent to the polynomial

$$f(x) = ax(x - t_1) \cdots (x - t_{n-1}) \pm k,$$

where $a, k, t_1, \cdots, t_{n-1}$ are positive integers, and the t's are distinct. We shall confine our attention to $f(x)$ and assume that $n \geqq 2$. We shall denote by x_0 a root of $f(x)$ whose absolute value is a minimum, and the other roots by $x_1, \cdots, x_{n-1}$. Taking the ratio of the coefficient of x to the constant term in each of the last two members of

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† For literature, see Dorwart and Ore, Annals of Mathematics, vol. 34 (1933), p. 81; A. Brauer, Jahresbericht der Deutscher Mathematiker Vereinigung, vol. 43 (1933), p. 124.

$$f(x) = ax \prod_{i=1}^{n-1} (x - t_i) \pm k = a \prod_{i=0}^{n-1} (x - x_i), \tag{1}$$

we have

$$\frac{at_1 \cdots t_{n-1}}{k} = \left| \frac{1}{x_0} + \cdots + \frac{1}{x_{n-1}} \right|.$$

Hence

$$|x_0| \leqq \frac{nk}{at_1 \cdots t_{n-1}}. \tag{2}$$

In the same way we infer from

$$f(x + t_j) = ax(x + t_j) \prod_{i=1, i\neq j}^{n-1} (x + t_j - t_i) \pm k$$
$$= a \prod_{i=0}^{n-1} (x + t_j - x_i),$$

that, to each index $j \geqq 1$, there corresponds an index p such that

$$| t_j - x_p | \leqq \frac{nk}{at_j \prod_{i=1, i\neq j}^{n-1} | t_j - t_i |}, \qquad (0 \leqq p \leqq n - 1). \tag{3}$$

THEOREM 1. *If the inequalities*

$$\begin{aligned} &2nk < at_1 \cdots t_{n-1}, \\ &2nk < at_j \prod_{i=1, i\neq j}^{n-1} | t_j - t_i |, \quad (j = 1, \cdots, n - 1), \end{aligned} \tag{4}$$

are satisfied, the roots of $f(x)$ are all real and lie within the intervals

$$\left[-\frac{1}{2}, +\frac{1}{2} \right], \quad \left[t_j - \frac{1}{2}, \ t_j + \frac{1}{2} \right], \quad (j = 1, \cdots, n - 1).$$

From (2), (3) and (4) we have

$$| x_0 | < \frac{1}{2}, \quad | t_j - x_p | < \frac{1}{2}, \quad (j = 1, \cdots, n - 1). \tag{5}$$

These inequalities show that each of the n circles

$$|x| = \frac{1}{2}, \qquad |x - t_j| = \frac{1}{2}, \qquad (j = 1, \cdots, n-1). \tag{6}$$

contains a root of $f(x)$. As $t_1, \cdots, t_{n-1}$ are distinct positive integers, no two of these circles intersect. It follows that each of the circles (6) contains one and only one root of $f(x)$. As a circle with center on the axis of reals which contains one of two conjugate imaginary numbers contains the other, while each of the circles (6) contains only one root of $f(x)$, the roots of $f(x)$ are real and lie within the stated intervals.* We shall choose our notation so that

$$|t_i - x_i| < \frac{1}{2}, \qquad (i = 1, \cdots, n-1). \tag{7}$$

3. *Irreducibility of $f(x)$.* It is convenient to define $\lambda = \lambda(n)$ by

$$\begin{aligned} &\lambda(2) = 1, \quad \lambda(3) = 4, \quad \lambda(4) = 6, \quad \lambda(5) = 3, \quad \lambda(6) = 1, \\ &\lambda(n) = 0 \text{ if } n \geqq 7. \end{aligned} \tag{8}$$

THEOREM 2. *The polynomial $f(x)$ is irreducible if at least one of the n inequalities*

$$a > 2^n k^2 + 1, \quad t_i > (3 + \lambda)k, \qquad (i = 1, \cdots, n-1), \tag{9}$$

is satisfied.

With the aid of (8) and the fact that the t's are distinct positive integers, it is readily proved that each of the inequalities (9) implies all of the inequalities (4). The roots of $f(x)$ are therefore isolated as described by Theorem 1.

Suppose that $f(x)$ is reducible:

$$f(x) = B(x)C(x) = \sum_{v=0}^{r} b_v x^{r-v} \cdot \sum_{v=0}^{s} c_v x^{s-v}, \qquad (b_0 c_0 \neq 0), \tag{10}$$

$(1 \leqq r \leqq n-1; 1 \leqq s \leqq n-1; r+s=n)$, the b's and c's being integers. Let $B(x)$ be that factor which has x_0 as a root; and let $x_1, \cdots, x_s$ be the roots of $C(x)$, so that

$$C(x) = c_0 \prod_{i=1}^{s} (x - x_i) = \sum_{v=0}^{s} c_v x^{s-v}. \tag{11}$$

* The referee has called my attention to an alternative proof which consists in showing that $f(1/2)$ and $f(-1/2)$ have opposite signs, and that $f(t_j - 1/2)$ and $f(t_j + 1/2)$ have opposite signs.

By Theorem 1, the roots of $C(x)$ are $>1/2$, and at most one of them is <1.

As $b_r c_s = \pm k$,

$$k \geqq \left| \frac{c_s}{c_0} \right| = x_1 \cdots x_s. \tag{12}$$

Hence

$$|x_j| \leqq 2k, \qquad (j = 1, \cdots, s). \tag{13}$$

Substituting $x = t_i$ in (10), we have

$$B(t_i)C(t_i) = f(t_i) = \pm k, \qquad (i = 1, \cdots, n-1).$$

Hence

$$| c_0 | \prod_{j=1}^{s} | t_i - x_j | = | C(t_i) | \leqq k.$$

As $|c_0| \geqq 1$, an index j exists such that

$$| t_i - x_j | \leqq k, \qquad (1 \leqq j \leqq s).$$

It follows from (13) that

$$t_i \leqq 3k, \quad (i = 1, \cdots, n-1). \tag{14}$$

Multiplying the equations

$$a x_j \prod_{i=1}^{n-1} | x_j - t_i | = k, \qquad (j = 1, \cdots, s),$$

obtained by substituting $x = x_j$ in (1), we have by (11),

$$a^s | c_s | \prod_{i=1}^{n-1} | C(t_i) | = k^s | c_0^n |. \tag{15}$$

From the nature of the roots of $C(x)$ and (12), we have $|c_s| \geqq |c_0|/2$. As c_s is a divisor of k, $|c_0| \leqq 2k$. As $C(t_i)$ is an integer $\neq 0$, it follows from (15) that $a^s \leqq 2^n k^{n+s-1}$. If s has its maximum value $n-1$, this inequality becomes $a^{n-1} \leqq 2^n k^{2(n-1)}$, whence $a \leqq 2^n k^2$. If $s < n-1$, we have, with the notation (7),

$$| t_i - x_j | > \frac{1}{2}, \qquad (i = s+1, \cdots, n-1).$$

(The right member may be replaced by 1 for all but one x_j.) Hence $|C(t_i)| > |c_0|/2$, and

$$\prod_{i=s+1}^{n-1} | C(t_i) | > \frac{| c_0^{n-s-1} |}{2^{n-s-1}}.$$

It follows from (15) that

$$a^s \leqq 2^{n-s}k^s | c_0 |^s \leqq 2^n k^{2s},$$

whence $a \leqq 2^n k^2$. As this inequality, and (14), contradict (9), we conclude that $f(x)$ is irreducible.

The example

$$\begin{aligned} b^2x(x-1)(x-3)(x-4) - 3b - 1 \\ = (bx^2 - 4bx + 3b + 1)(bx^2 - 4bx - 1), \end{aligned}$$

in which $a = b^2$, $\pm k = 3b+1$, shows that the first of the inequalities (9) cannot be replaced by one which is linear in k.

While the inequalities (9) can undoubtedly be weakened by further analysis, without affecting the irreducibility of $f(x)$, they suffice to establish the following general theorems.

THEOREM 3. *Only a finite number of non-equivalent reducible polynomials of degree n exist which assume a given integral value* $\neq 0$ *for n different integral values of the variable.*

For, if n and k are fixed positive integers, only a finite number of sets of positive integers $a, t_1, \cdots, t_{n-1}$ exist which violate all the inequalities (9).

THEOREM 4. *If k is a fixed integer* $\neq 0$, *and n is sufficiently large, every polynomial of degree n which assumes the value k for n distinct integral values of its argument is irreducible.*

At least one of the integers $t_1, \cdots, t_{n-1}$ is $\geqq n-1$. Hence if $n \geqq 7$ and $n > 3k+1$, at least one of the inequalities

$$t_i > (3 + \lambda)k, \qquad (i = 1, \cdots, n-1),$$

is satisfied, and $f(x)$ is irreducible.

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