A NOTE ON THE JACOBI CONDITION FOR PARAMETRIC PROBLEMS IN THE CALCULUS OF VARIATIONS*

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1. *Introduction.* An elegant treatment of the Jacobi condition for parametric problems in $(y_1 \cdots y_n)$-space has been given by Bliss.† He defines conjugate points in terms of solutions η_i of the Jacobi equations which satisfy the relations $y_i'\eta_i=0$. With the help of these solutions he establishes criteria for conjugate points in terms of the general solutions of the Jacobi equations. Since there are but $2n-2$ linearly independent solutions η_i satisfying the conditions $y_i'\eta_i=0$, the treatment given by Bliss is quite different from that usually given for non-parametric problems. It is well known that the methods of Bliss are still applicable if the equation $y_i'\eta_i=0$ is replaced by an equation such as $y_i'\eta_i'=0$, $y_i'\eta_i'=$constant, and others.‡ In the present paper we use the equation $y_i'\eta_i'=$constant in defining conjugate points and obtain the same results as Bliss. The method used is, however, quite different from that of Bliss and has the advantage that it is almost identical with that usually given for the non-parametric problems. This follows because there are $2n$ linearly independent solutions of the type considered in this paper. Other equations for which the method here used is still applicable are also discussed.

2. *The Necessary Condition of Jacobi.* The problem to be considered is that of minimizing an integral

$$I = \int_{t_1}^{t_2} f(y_1, \cdots, y_n, y_1', \cdots, y_n')dt = \int_{t_1}^{t_2} f(y, y')dt$$

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* Presented to the Society, June 23, 1933.

† Bliss, *Jacobi's condition for problems of the calculus of variations in parametric form,* Transactions of this Society, vol. 17 (1916), pp. 195–206.

‡ Graves, *Discontinuous solutions in space problems of the calculus of variations,* American Journal of Mathematics, vol. 52 (1930), p. 17. See also Wren, *A new theory of parametric problems in the calculus of variations,* Contributions to the Calculus of Variations, 1930, The University of Chicago Press, pp. 175–185.

in a class of arcs $y_i = y_i(t)$, $(t_1 \leqq t \leqq t_2;\ i = 1, \cdots, n)$, which join two fixed points 1 and 2 in $(y_1 \cdots y_n)$-space. The hypotheses upon which our analysis is based are those of Bliss.* The notations are also those of Bliss except that we use the tensor analysis summation convention instead of matrix notation.

A *non-singular* arc is one whose determinant

$$\begin{vmatrix} f_{y_i' y_k'} & y_i' \\ y_k' & 0 \end{vmatrix}$$

is different from zero at each element (y, y') on it A non-singular minimizing arc E_{12} without corners is always an extremal arc. Moreover, the parameter t can be chosen so that the functions $y_i(t)$ defining E_{12} have continuous third derivatives.†

We define a function $2\omega(t, \eta, \eta')$ to be the function $\Omega(\eta, \eta')$ used by Bliss. Along a non-singular minimizing arc E_{12} without corners the second variation of the integral I is expressible in the form

$$I_2 = \int_{t_1}^{t_2} 2\omega(t, \eta, \eta')dt,$$

and this expression must be $\geqq 0$ for every set of admissible variations $\eta_i(t)$ which vanish at t_1 and t_2.

The *Jacobi equations* are the equations

$$J_i(\eta) = (d/dt)\omega_{\eta_i'} - \omega_{\eta_i} = 0.$$

They satisfy the relations $y_i' J_i(\eta) \equiv 0$ and hence are not independent. They are satisfied identically by the functions $\eta_i = \rho(t) y_i'$, where $\rho(t)$ is an arbitrary function possessing continuous first derivatives.‡

A *special solution* η_i of the Jacobi equations is defined to be one which satisfies identically the equations $y_i' \eta_i' = \text{constant}$. Every special solution satisfies with $\lambda \equiv 0$ the equations

$$(1) \qquad J_i(\eta) + \lambda y_i' = 0, \qquad y_i' \eta_i'' + y_i'' \eta_i' = 0.$$

Conversely every solution η_i, λ of these equations has $y_i' \eta_i' = \text{constant}$, and $\lambda \equiv 0$ on account of the relations $y_i' J_i(\eta) \equiv 0$,

* Loc. cit., p. 196.

† Bliss, loc. cit., pp. 197–198.

‡ Bliss, loc. cit., p. 201.

and hence the functions η_i define special solutions of the Jacobi equations. Since E_{12} is non-singular, equations (1) can be solved for the variables η'', λ. In the solutions the values for λ and η_i'' have the form $\lambda \equiv 0$, since $y_i' J_i(\eta) \equiv 0$, and

$$(2) \qquad \eta_i'' = A_{ik}(t)\eta_k + B_{ik}(t)\eta_k'.$$

A solution η_i of the Jacobi equations is therefore a special solution if and only if it also satisfies equations (2). The usual existence theorems for differential equations now tell us that through each element $(t_0, \eta_{i0}, \eta_{i0}')$ there passes one and only one special solution η_i. Hence there exist $2n$ linearly independent special solutions.

A point 3 is said to be *conjugate* to the point 1 on E_{12} if there exists a special solution $\eta_i = u_i$ of the Jacobi equations whose functions $u_i(t)$ vanish at t_1 and t_3 but are not identically zero on t_1t_3. With this definition in mind the following necessary condition can be proved readily by the methods of Bliss.*

THEOREM 1. THE NECESSARY CONDITION OF JACOBI. *On a non-singular minimizing arc E_{12} without corners there can be no point 3 conjugate to 1 between 1 and 2.*

In the proof of this theorem it is found by the methods of Bliss that if there were a special solution u_i defining a point 3 conjugate to 1 between 1 and 2 on E_{12}, then this solution would take the values $u_i = 0$, $u_i' = cy_i'$ at $t = t_3$. There is but one special solution taking these values at $t = t_3$, namely, the solution $u_i = \phi'(a\phi + b)y_i'$, where $a\phi'^2(t_3) = c$, $b = -a\phi(t_3)$, and

$$(3) \qquad \phi(t) = \int_{t_1}^{t} dt/(y_i' y_i')^{1/2},$$

as is readily verified by substitution. This solution, however, vanishes at $t = t_1$ only in case $a = b = c = 0$, that is, only in case $u_i \equiv 0$. It follows that there can be no point 3 conjugate to 1 between 1 and 2 on E_{12}, as was to be proved.

3. *The Determination of Conjugate Points.* Let E_{12} be a non-singular extremal arc whose parameter t has been chosen so that the functions $y_i(t)$ defining E_{12} have continuous third derivatives.

* Loc. cit., p. 200.

LEMMA 1. *A set of* $2n$ *special solutions* u_{is}, $(s=1, \cdots, 2n)$, *of the Jacobi equations have a determinant*

$$\begin{vmatrix} u_{is}(t) \\ u_{is}'(t) \end{vmatrix} \tag{4}$$

which is either identically zero or else everywhere different from zero.

This result is a consequence of well known theorems on linear homogeneous differential equations of the second order applied to equations (2). A further consequence of these theorems is that every special solution u_i is expressible linearly with constant coefficients in terms of $2n$ special solutions u_{is} whose determinant (4) is different from zero. The following two lemmas can now be established by the usual methods.*

LEMMA 2. *The points* 3 *conjugate to* 1 *on* E_{12} *are determined by the zeros* $t_3 \neq t_1$ *of the determinant*

$$\Delta(t, t_1) = \begin{vmatrix} u_{is}(t) \\ u_{is}(t_1) \end{vmatrix} \tag{5}$$

formed for $2n$ *special solutions* u_{is} *of the Jacobi equations whose determinant* (4) *is different from zero.*

LEMMA 3. *If the functions* $u_{ik}(t)$, $(k=1, \cdots, n)$, *define* n *linearly independent special solutions of the Jacobi equations which vanish at* $t=t_1$, *then the points* 3 *conjugate to* 1 *on* E_{12} *are determined by the zeros* $t_3 \neq t_1$ *of the determinant* $|u_{ik}(t)|$.

We also need the further lemma:

LEMMA 4. *Every solution* η_i *of the Jacobi equations has associated with it a two-parameter family of special solutions*

$$u_i = \eta_i - \rho y_i',$$

where

$$\rho = \phi'(a\phi + b + \int_{t_1}^{t} y_i' \eta_i' \phi' dt), \tag{6}$$

$\phi(t)$ *is the function* (3), *and* a, b *are arbitrary constants.*

* See, for example, Bliss, *The problem of Lagrange in the calculus of variations*, American Journal of Mathematics, vol. 52 (1930), pp. 727–728.

This result is readily obtained by solving for ρ in the equations

$$y_i' u_i' = y_i' \eta_i' - \rho' y_i' y_i' - \rho y_i'' y_i' = -a.$$

From Lemma 4, with $a=b=0$, it follows that every set of $2n-2$ solutions u_{iq}, $(q=1, \cdots, 2n-2)$, of the Jacobi equations has associated with it a set of $2n-2$ two-parameter families of special solutions $v_{iq}=u_{iq}-\rho_q y_i'$. Moreover, the functions $\phi\phi' y_i'$, $\phi' y_i'$, where ϕ is the function (3), also define special solutions, as is easily seen by substitution. The determinants (4) and (5) formed for the $2n$ special solutions v_{iq}, $\phi\phi' y_i'$, $\phi' y_i'$ are equal, respectively, to

$$\phi'^3 d(t), \qquad \phi(t)\phi'(t)\phi'(t_1)D(t, t_1),$$

where

$$d(t) = \begin{vmatrix} u_{iq}(t) & 0 & y_i'(t) \\ u_{iq}'(t) & y_i'(t) & y_i''(t) \end{vmatrix},$$

$$D(t, t_1) = \begin{vmatrix} u_{iq}(t) & y_i'(t) & 0 \\ u_{iq}(t_1) & 0 & y_i'(t_1) \end{vmatrix}.$$

Since $\phi' > 0$, the zeros of $D(t, t_1)$ are identical with those of $\Delta(t, t_1)$. Hence, by the use of Lemmas 1 and 2, we obtain the following theorem of Bliss.

THEOREM 2. *The determinant $d(t)$ formed for $2n-2$ solutions u_{iq}, $(q=1, \cdots, 2n-2)$, of the Jacobi equations is either identically zero or else everywhere different from zero. Moreover, the points 3 conjugate to 1 on E_{12} are determined by the zeros $t_3 \neq t_1$ of the determinant $D(t, t_1)$ formed for $2n-2$ solutions u_{iq} whose determinant $d(t)$ is different from zero.*

We can also prove the further criterion for conjugate points given by Bliss.

THEOREM 3. *If the functions $u_{ir}(t)$, $(r=1, \cdots, n-1)$, define $n-1$ solutions of the Jacobi equations whose matrix*

$$\left\| \begin{matrix} u_{ir} & 0 & y_i' \\ u_{ir}' & y_i' & y_i'' \end{matrix} \right\| \tag{7}$$

has rank $n+1$ at $t=t_1$, while the determinant

$$D(t) = \left| u_{ir}(t) \quad y_i'(t) \right|$$

has rank 1 *at* $t=t_1$, *then the points* 3 *conjugate to* 1 *on* E_{12} *are determined by the zeros* $t_3\neq t_1$ *of* $D(t)$.

In order to prove this we note that since $D(t_1)$ has rank 1, there exist constants c_r such that $u_{ir}-c_r y_i'=0$ at $t=t_1$. By taking $a=0$ and $b=c_r/\phi'(t_1)$ in Lemma 4, it is found that the $n-1$ solutions u_{ir} have associated with them $n-1$ special solutions $v_{ir}=u_{ir}-\rho_r y_i'$ such that $\rho_r(t_1)=c_r$ and hence such that $v_{ir}(t_1)=0$. As above, it is easily seen by substitution that the functions $v_i=\phi'\phi y_i'$ define a special solution having $v_i(t_1)=0$. The n special solutions v_{ir}, v_i are linearly independent since the matrix (7) has rank $n+1$. Moreover, the determinant $|v_{ir}\ v_i|$ is equal to $\phi\phi' D(t)$. Its zeros, therefore, coincide with those of $D(t)$ since $\phi'(t)>0$. The theorem now follows from Lemma 3.

4. *Other Types of Special Solutions*. In §§2 and 3 above, the equation $y_i'\eta_i'=$ constant was used in order to define special solutions. We could use instead any equation which is equivalent to one of the form

$$y_i'\eta_i'' + C_i(t)\eta_i' + D_i(t)\eta_i = 0$$

and which has a two-parameter family of solutions of the form $\eta_i=[a\rho_1(t)+b\rho_2(t)]y_i'(t)$ with the property that for every pair of constants $(a, b)\neq(0, 0)$ the expression $a\rho_1+b\rho_2$ vanishes at most once on the interval under consideration. The equations $y_i'\eta_i=at+b$ and $y_i'\eta_i'=ay_i'y_i'$, for example, have this property, as one readily verifies.

It is interesting to note that the special solutions used in §§ 2 and 3 above are closely related to extremal families having arc length as their parameter. If the functions

$$y_i = y_i(t, c_1, \cdots, c_{2n-2})$$

define a $(2n-2)$-parameter family of extremals containing E_{12} for $c_q=c_{q0}$, $(q=1, \cdots, 2n-2)$, and having arc length as parameter, then along E_{12} the derivatives y_{ia}, y_{ib}, y_{ic_q} of the functions $y_i(at+b, c_q)$ form $2n$ special solutions having the properties described in Lemmas 1 and 2. In order to prove that these derivatives define special solutions one needs only to differentiate the identity $y_i'y_i'=a^2$ with respect to a, b, c_q and set $c_q=c_{q0}$.

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