LAGRANGE MULTIPLIERS FOR FUNCTIONS OF INFINITELY MANY VARIABLES*

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The purpose of this note is to extend the Lagrange multiplier theorem to the case of a maximum of a function of infinitely many variables subject to an infinity of auxiliary conditions. The underlying implicit function theorems used are due to Hart.† The proof employs two lemmas on normal determinants and associated linear systems of equations which seem to have been overlooked.‡ One of these incidentally renders one assumption in Hart's implicit function theorem redundant.

LEMMA 1. *If* $\sum_{i,k}|a_{ik}|=A$ *and* A_{ik} *is the minor of* $\delta_{ik}+a_{ik}$ *in the determinant* $\Delta=|\delta_{ik}+a_{ik}|$, *then* $\sum_{i,k}|A_{ik}|$ $(i\neq k)$ *converges and the* $|A_{ii}|$ *are bounded.*

PROOF. Since $\sum_{i,k}|a_{ik}|$ converges, $\prod_k(1+\sum_i|a_{ik}|)$ converges. If $p=a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ni_1}$, the infinite product $\prod(1+|p|)$ extended over all values of p is dominated by the product $\prod_k(1+\sum_i|a_{ik}|)$ and converges to a value P. A term of A_{ik}, $(i\neq k)$, has one of the forms

$$a_{ki}T, \qquad a_{ki_1}a_{i_1i_2}\cdots a_{i_ni}T,$$

where T is a product of factors p and the indices are all distinct. Hence

$$|A_{ik}|\leqq P\left\{|a_{ki}|+\sum_n\sum_{i_1\cdots i_n}|k,i_1,i_2,\cdots,i_n;i|\right\},$$

where $|k,i_1,i_2,\cdots,i_n;i|$ is $|a_{ki_1},a_{i_1i_2},\cdots,a_{i_ni}|$ or zero according as the indices are distinct or not. Now

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† W. L. Hart, *Differential equations and implicit functions in infinitely many variables*, Transactions of this Society, vol. 18 (1917), Theorems XII, XIII, VI.

‡ For the normal determinant theory, see F. Riesz, *Les Systèmes d'Equations Linéaires . . .*, 1913.

$$\sum_{k,i}\sum_{i_1,\cdots,i_n} |k, i_1, \cdots, i_n; i|$$
$$\leqq \sum_{k,i_1}|a_{ki_1}|\sum_{i_1i_2}|a_{i_1i_2}| \cdots \sum_{i_ni}|a_{i_ni}| = A^{n+1}.$$

Let us assume $A < 1$. Then

$$\sum_{i,k}|A_{ik}| \leqq P\left\{\sum_{i,k}|a_{ki}| + \sum_{i,k}\sum_{n}\sum_{i_1\cdots i_n}|k, i_1, \cdots, i_n; i|\right\}$$

$$\leqq P\left\{A + \sum_{n} A^{n+1}\right\} = \frac{PA}{1-A}\cdot$$

It is clear that $|A_{ii}| \leqq P$. The restriction $A < 1$ may be removed by use of a convergence factor.*

LEMMA 2. *If* $\sum_{i,k}|a_{ik}| = A$, $\sum_i |b_i| = B$, *and* $\Delta \neq 0$, *then the solution* x_k *of* $x_i + \sum_k a_{ik}x_k = b_i$ *is such that* $\sum_k |x_k|$ *converges.*

PROOF. The solution is given by

$$x_k = \frac{1}{\Delta}\sum_i A_{ik}b_i = \frac{1}{\Delta}\left\{\sum_{i\neq k} A_{ik}b_i + A_{kk}\,b_k\right\}.$$

Then

$$\sum_k |x_k| \leqq \frac{1}{|\Delta|}\sum_k\left\{\sum_{i\neq k}|A_{ik}b_i| + |A_{kk}b_k|\right\}$$
$$\leqq \frac{B}{|\Delta|}\sum_k\sum_{i\neq k}|A_{ik}| + \frac{P}{|\Delta|}\sum_k |b_k|,$$

which, with Lemma 1, proves Lemma 2.

Let S, T be the sets of points ξ, η with coordinates x_i, y_i such that $|x_i - a_i| < r_i < r$, $|y_i - b_i| < r_i' < r$, $(i = 1, 2, \cdots)$, respectively, and let R be the space (ξ, η). A function $f(\xi, \eta)$ is called completely continuous in R if to every $\epsilon > 0$ there is a $\delta_\epsilon > 0$ such that $|f(\xi', \eta') - f(\xi'', \eta'')| < \epsilon$ when $|x_i' - x_i''| < \delta_\epsilon$, $|y_i' - y_i''| < \delta_\epsilon$, and (ξ', η'), (ξ'', η'') are in R. The complete continuity of $f(\xi)$, $f(\eta)$ in S, T, respectively, is similarly defined.

IMPLICIT FUNCTION THEOREM. *If*

(1) $\phi_i(\xi, \eta)$ *and* $\partial\phi_i/\partial y_j$, $(i, j = 1, 2, \cdots)$, *are completely continuous in* R;

* See F. Riesz, loc. cit., p. 38.

(2) $|\phi_i(\xi, \eta)| \leqq M_i \leqq M$ *in* R;
(3) $\sum_{i,j} |\delta_{ij} - \partial\phi_i/\partial y_j|$ *converges uniformly in* R;
(4) *the determinant* $|\partial\phi_i/\partial y_j|$ *is not zero at* (α, β) *with coordinates* a_i, b_i;
(5) $\phi_i(\xi, \eta) = 0$ *in* R;
then unique solutions $y_i(\xi)$ *of the system* $\phi_i = 0$ *exist in a neighborhood of* (α, β) *and are completely continuous in a neighborhood of* α.*

If
(6) $\partial\phi_i/\partial x_k$ *are completely continuous in* R;
(7) $|\partial\phi_i/\partial x_k| \leqq N_k \leqq N$ *in* R;
then $\partial y_i/\partial x_k$ *exist in a neighborhood of* α, *the equations*

$$\frac{\partial\phi_i}{\partial x_k} + \sum_j \frac{\partial\phi_i}{\partial y_j}\frac{\phi y_j}{\partial x_k} = 0$$

are satisfied, and $|\partial y_i/\partial x_k| < Q_k$ *in the neighborhood.*

Differentiation Theorem. *If* $f(\eta)$ *and* $\partial f/\partial y_i$ *are completely continuous in* T;
(8′) $\sum_j |\partial f/\partial y_j|$ *converges uniformly in* T;
(9′) $y_j(v)$, dy_j/dv *are continuous in* $a \leqq v \leqq b$ *and* $\eta(v)$ *lies in* T;
(10′) $\sum_j |(\partial f/\partial u_j)\cdot(\partial y_j/\partial v)|$ *converges uniformly in* $a \leqq v \leqq b$, y_j *in* T; *then if* $G(v) = f[\eta(v)]$,

$$\frac{dG}{dv} = \sum_i \frac{\partial f}{\partial y_i}\frac{dy_i}{dv}.$$

We now state the Lagrange multiplier theorem.

If $\phi_i(\xi, \eta)$ *have the properties* (1), $\cdots$, (7); $f(\xi, \eta)$, $\partial f/\partial x_k$, $\partial f/\partial y_k$ *are completely continuous in* R *and* $f(\xi, \eta)$ *has a maximum at* (α, β) *subject to the conditions* $\phi_i(\xi, \eta) = 0$ *in* R;
(8) $\sum_i \partial f/\partial y_i$ *converges uniformly in* R;
then there exist λ_i *such that* $\sum_i |\lambda_i|$ *converges and, at* (α, β),

$$\frac{\partial f}{\partial y_k} + \sum_i \lambda_i \frac{\partial\phi_i}{\partial y_k} = 0, \tag{9}$$

$$\frac{\partial f}{\partial x_k} + \sum_i \lambda_i \frac{\partial\phi_i}{\partial x_k} = 0, \quad (k = 1, 2, \cdots). \tag{10}$$

* From Lemma 1 it follows that the assumption that $\sum_k |D_{ki}|$ is bounded in i, where D_{ki} are the minors of the Jacobian of the system $\phi_i = 0$, is redundant in Hart's Theorem XII.

PROOF. From conditions (3), (4), (8) and Lemma 2, it follows that (9) has a solution λ_i with $\sum_i |\lambda_i| = \lambda$ at (α, β). The system $\phi_i = 0$ has solutions $y_i(\xi)$. Condition (8′) follows from (8); (9′) from (1), $\cdots$, (7); and (10′) from (6), (7), (8). Hence at the maximum

$$\frac{\partial f}{\partial x_k} + \sum_j \frac{\partial f}{\partial y_j}\frac{\partial y_j}{\partial x_k} = 0. \tag{11}$$

We note that

$$\begin{aligned}
\sum_{i,j}\left|\lambda_i \frac{\partial \phi_i}{\partial y_j}\frac{\partial y_j}{\partial x_k}\right| &\leqq Q_k \sum_{i,j}\left|\lambda_i \frac{\partial \phi_i}{\partial y_j}\right| \\
&\leqq Q_k\left\{\sum_{i,j} |\lambda_i| \left|\frac{\partial \phi_i}{\partial y_j} - \delta_{ij}\right| + \sum_{i,j} |\lambda_i \delta_{ij}|\right\} \\
&\leqq Q_k\left\{\lambda \sum_{i,j}\left|\frac{\partial \phi_i}{\partial y_j} - \delta_{ij}\right| + \lambda\right\},
\end{aligned} \tag{12}$$

and the convergence follows from (3). From (7) and (12) we have

$$\begin{aligned}
&\sum_i \lambda_i \frac{\partial \phi_i}{\partial x_k} + \sum_i \lambda_i \sum_j \frac{\partial \phi_i}{\partial y_j}\frac{\partial y_j}{\partial x_k} \\
&\qquad = \sum_i \lambda_i \frac{\partial \phi_i}{\partial x_k} + \sum_j \frac{\partial y_j}{\partial x_k}\sum_i \lambda_i \frac{\partial \phi_i}{\partial y_j} = 0.
\end{aligned} \tag{13}$$

Combining (11), (13), (9), we have

$$\begin{aligned}
&\frac{\partial f}{\partial x_k} + \sum_j \frac{\partial f}{\partial y_j}\frac{\partial y_j}{\partial x_k} + \sum_i \lambda_i \frac{\partial \phi_i}{\partial x_k} + \sum_j \frac{\partial y_j}{\partial x_k}\sum_i \lambda_i \frac{\partial \phi_i}{\partial y_j} \\
&\quad = \frac{\partial f}{\partial x_k} + \sum_i \lambda_i \frac{\partial \phi_i}{\partial x_k} + \sum_j \frac{\partial y_j}{\partial x_k}\left[\frac{\partial f}{\partial y_j} + \sum_i \lambda_i \frac{\partial \phi_i}{\partial y_j}\right] \\
&\quad = \frac{\partial f}{\partial x_k} + \sum_i \lambda_i \frac{\partial \phi_i}{\partial x_k} = 0,
\end{aligned} \tag{14}$$

which is (10), and the theorem is proved.

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