A MODULAR MANIFOLD ASSOCIATED WITH THE GENERALIZED KUMMER MANIFOLD ($p=3$)

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1. *Introduction.* The generalized Kummer p-way manifold, K_p, is determined by equating the homogeneous coordinates of a point P in S_{2^p-1} to linearly independent theta functions of the second order and zero characteristic.* As the variables u in these functions change, the point $P(u)$ runs over the manifold K_p in S_{2^p-1}. If the variables u be increased by a half-period π, the point $P(u)$ is transformed into the point $P(u')=P(u+\pi)$. Thus the half-periods determine a group $G_{2^{2p}}$ of birational transformations of K_p into itself. Klein and Wirtinger have shown that these birational transformations are effected on K_p by the operations of a collineation group $G_{2^{2p}}$ in S_{2^p-1} under which K_p is invariant. The functions which define the position of P may be so chosen that the coefficients of the collineations of $G_{2^{2p}}$ are *numerical.* A convenient choice† is that of functions $Z_{\eta_1\ldots\eta_p}$, ($\eta_i=0, 1$), for which the addition of a particular half-period π_{η_i}

* The reader is referred to the following sources.

E. Bertini, *Einführung in die Projektive Geometrie Mehrdimensionaler Räume*, 1924.

A. B. Coble, *An application of finite geometry to the characteristic theory of the odd and even theta functions*, Transactions of this Society, vol. 14 (1913), pp. 241–276.

A. B. Coble, *An isomorphism between theta characteristics and the* ($2p+2$)-*point*, Annals of Mathematics, (2), vol. 17 (1916), pp. 101–112.

A. B. Coble, *Algebraic Geometry and Theta Functions*, Colloquium Publications of this Society, 1929.

R. W. H. T. Hudson, *Kummer's Quartic Surface*, 1905.

Felix Klein, *Theorie der Elliptischen Modulfunktionen*, 1890.

A. Krazer, *Lehrbuch der Thetafunktionen*, 1903.

F. Schottky, *Beziehungen zwischen den sechzehn Thetafunctionen von zwei Variabeln*, Journal für Mathematik, vol. 105 (1889), pp. 233–249.

H. Stahl, *Theorie der Abel'schen Functionen*, 1896.

W. Wirtinger, *Über eine Verallgemeinerung der Theorie der Kummer'schen Fläche and ihrer Beziehungen zu den Thetafunctionen zweier Variabeln*, Monatshefte, vol. 1 (1890).

† Coble, Colloquium, loc. cit., p. 94.

changes $Z_{\eta_1 \ldots \eta_p}$ into $(-1)^{\eta_i} Z_{\eta_1 \ldots \eta_p}$; and for which the addition of another particular half-period π_{η_i} interchanges the values 0, 1 of the index η_i of $Z_{\eta_1 \ldots \eta_p}$, $(i=1, \cdots, p)$. The 2^{2p} points $P(u)$ for which $u=\pi$ are singular points of K_p. A particular K_p is determined when the group $G_{2^{2p}}$ of K_p, and a singular point of K_p, are given.

When $G_{2^{2p}}$ is fixed, say in the simple form just indicated, there is a family, F_p, of K_p's each of which admits this group. This family is obtained by variation of the moduli a_{ij} of the theta functions. As K_p runs through this family F_p, the 2^{2p} singular points of K_p describe a locus in S_{2^p-1}, the *modular manifold* M, with which we shall be concerned for the case $p=3$. In the case $p=1$, K_p is a doubly covered S_1 with four branch points as singular points. As the modulus a of the elliptic thetas changes, these branch points run over the entire S_1. In the case $p=2$, K_p is the Kummer surface in S_3 with 16 nodal singular points. As the three moduli a_{ij} change, these nodes run over the entire S_3. In the case $p=3$, K_p is the Kummer 3-way in S_7 of order 24 with 64 four-fold singular points. In this case, however, as the *six* moduli a_{ij} change, the singular points in S_7 run over a manifold of dimension six and order 16, M_6^{16}. The purpose of this article is to discuss this manifold M_6^{16} which appears from the transcendental view-point, with respect to its projective and group-theoretic properties. We use without further explanation the standard notations of the theta function theory. The equation of M_6^{16} is obtained in §2. In §§3–4, the multiplicities of certain loci on M_6^{16} and on its sections are determined. In §5 a projective determination of M_6^{16} by means of these multiplicities is indicated.

2. *Determination of the Equation of* M_6^{16}. The equation of M_6^{16} may be obtained from one of the relations existing between products of the zero values of the even thetas. One such is

$$\vartheta_{lmn7}\vartheta_{lmn8}\vartheta_{iln7}\vartheta_{iln8} \pm \vartheta_{jmn7}\vartheta_{jmn8}\vartheta_{ijn7}\vartheta_{ijn8} \pm \vartheta_{kmn7}\vartheta_{kmn8}\vartheta_{ikn7}\vartheta_{ikn8} = 0.$$

This may be written, in a particular case, as

$$(1) \qquad \sum_{\alpha=1}^{3} \pm \left(c_{\alpha457}^2 c_{\alpha458}^2 c_{\alpha467}^2 c_{\alpha468}^2\right)^{1/2} = 0,$$

where $\vartheta_{ijkl}(0)=c_{ijkl}$. To obtain the expression for this in terms of the modular functions $z_{ijk}=z_{ijk}(0)$, it is convenient to write the relation as $(a)^{1/2}+(b)^{1/2}+(c)^{1/2}=0$. The rational form is $c^2-2c(a+b)+(a-b)^2=0$. In expressing the c_{ijkl}^2 in terms of the z_{ijk}'s it is convenient to set $z_{000}=z$, and the remaining $z_{ijk}=z_1, z_2, \cdots, z_7$, where the $z_1, z_2, \cdots, z_7$ are thought of as attached to the seven points of a finite planar geometry, mod 2, say $PG(2, 2)$, in particular z_{ijk} attaching to the point with coordinates $i, j, k=0, 1$. Then the a, b, c, above, become

$$
\begin{aligned}
a = {} & \left[(z^2 - z_6^2) + (z_1^2 - z_2^2) + (z_3^2 - z_7^2) + (z_5^2 - z_4^2)\right] \\
& \cdot \left[(z^2 - z_6^2) - (z_1^2 - z_2^2) - (z_3^2 - z_7^2) + (z_5^2 - z_4^2)\right] \\
& \cdot \left[(z^2 - z_6^2) + (z_1^2 - z_2^2) - (z_3^2 - z_7^2) - (z_5^2 - z_4^2)\right] \\
& \cdot \left[(z^2 - z_6^2) - (z_1^2 - z_2^2) + (z_3^2 - z_7^2) - (z_5^2 - z_4^2)\right], \\
b = {} & \left[(z^2 + z_6^2) - (z_1^2 + z_2^2) + (z_3^2 + z_7^2) - (z_5^2 - z_4^2)\right] \\
& \cdot \left[(z^2 + z_6^2) + (z_1^2 + z_2^2) - (z_3^2 + z_7^2) - (z_5^2 + z_4^2)\right] \\
& \cdot \left[(z^2 + z_6^2) - (z_1^2 + z_2^2) - (z_3^2 + z_7^2) + (z_5^2 - z_4^2)\right] \\
& \cdot \left[(z^2 + z_6^2) + (z_1^2 + z_2^2) + (z_3^2 + z_7^2) + (z_5^2 + z_4^2)\right], \\
c = {} & \left[2zz_6 + 2z_1z_2 + 2z_3z_7 + 2z_4z_5\right] \cdot \left[2zz_6 + 2z_1z_2 - 2z_3z_7 - 2z_4z_5\right] \\
& \cdot \left[2zz_6 - 2z_1z_2 + 2z_3z_7 - 2z_4z_5\right] \cdot \left[2zz_6 - 2z_1z_2 - 2z_3z_7 + 2z_4z_5\right].
\end{aligned}
$$

The rationalized form then yields the following equation for M_6^{16}:

$$
\begin{aligned}
M_6^{16} = {} & zz_1z_2z_3z_4z_5z_6z_7 \Big\{ -2z^8 + 4z^4 \sum_7 z_1^4 - 16z^2 \sum_7 z_1^2 z_2^2 z_6^2 \\
& - 2 \sum_7 z_1^8 - 16 \sum_7 z_3^2 z_4^2 z_5^2 z_7^2 + 4 \sum_{21} z_1^4 z_2^4 \Big\} \\
& + z^8 \Big(\sum_7 z_3^2 z_4^2 z_5^2 z_7^2 \Big) + z^6 \Big(- \sum_{7\cdot 4} z_1^4 z_2^2 z_3^2 z_4^2 \Big) \\
& + z^4 \Big(- \sum_{7\cdot 4} z_3^6 z_4^2 z_5^2 z_7^2 + 2 \sum_{7\cdot 3} z_6^4 z_3^2 z_4^2 z_5^2 z_7^2 + \sum_{28} z_1^4 z_2^4 z_3^4 \Big) \\
& + z^2 \Big(\sum_{7\cdot 4} z_6^8 \cdot z_1^2 z_3^2 z_5^2 - \sum_{7\cdot 4\cdot 3} z_6^4 z_1^6 z_3^2 z_5^2 + 2 \sum_{7\cdot 6} z_6^4 z_7^4 z_1^2 z_3^2 z_5^2
\end{aligned}
\tag{2}
$$

$$+ 72z_1^2 z_2^2 z_3^2 z_4^2 z_5^2 z_6^2 z_7^2 \Big) + \Big(\sum_{7\cdot 3} z_6^8 \cdot z_3^2 z_4^2 z_5^2 z_7^2 - \sum_{7\cdot 4\cdot 3} z_6^6 z_2^2 z_1^4 z_3^2 z_5^2$$

$$+ \sum_{7\cdot 4} z_6^4 \cdot z_1^4 z_3^4 z_5^4 + 2 \sum_{21} z_6^4 z_2^4 \cdot z_3^2 z_4^2 z_5^2 z_7^2 \Big) = 0.$$

This final form of the equation of M_6^{16} contains first the odd and then the even powers of z in descending order. The symbol $\sum_7 z_1^2 z_2^2 z_6^2$ represents the sum of seven terms whose three subscripts lie on a line in the finite geometry,* $PG(2, 2)$. Other symbols in (2) refer to the seven quadrilaterals, seven points with each of four outside lines, seven points with each of three outside quadrangles, twenty-eight triangles, and twenty-one pairs of intersecting lines. Thus the modular locus is of order 16, with a 7-fold point at each reference point with a tangent septimic cone made up of the seven reference S_6's on the 7-fold point, and with similar behavior at the conjugates of these under the group of M_6^{16}.

This modular manifold, M_6^{16}, is invariant† under a group G of order $8!\cdot 36\cdot 64$ generated by elements $J_{abc,lmn}$. The situation is given by the following statement.

THEOREM 1. *M_6^{16} is invariant under a group G of order* $8!\cdot 36\cdot 64$. *This group G is the largest collineation group which contains the collineation group G_{64} of K_3^{24} as an invariant subgroup.*

Corresponding to the fact that the seven points of a finite plane (mod 2) may be permuted in 168 ways without destroying the linearity of triads, the equation of M_6^{16} is invariant under a permutation group of $z_1, \cdots, z_7$ of order 168, a subgroup of G. There exists a larger permutation subgroup $G_{8\cdot 168}$ on $z, z_1, \cdots, z_7$. Under the symmetry imposed by this larger subgroup in the equation (2) of M_6^{16} is reduced to the form

$$M_6^{16} = \Big\{ \prod_8 (z) \Big\} \Big\{ -2\sum_8 z^8 + 4 \sum_{28} z^4 z_1^4 - 16 \sum_{14} z^2 z_1^2 z_2^2 z_6^2 \Big\}$$

$$(3) \quad + \sum_{7\cdot 8} z^8 z_3^2 z_4^2 z_5^2 z_7^2 - \sum_{7\cdot 4\cdot 8} z^6 z_1^4 z_2^2 z_3^2 z_4^2 + 2 \sum_{7\cdot 12} z^4 z_6^4 z_3^2 z_4^2 z_5^2 z_7^2$$

$$+ \sum_{7\cdot 8} z^4 z_1^4 z_2^4 z_3^4 + 72 \Big\{ \prod_8 (z) \Big\}^2 = 0.$$

* Coble, Colloquium, loc. cit., p. 105.

† Coble, Colloquium, loc. cit., pp. 100–101, and p. 107.

3. *Multiple Loci of* M_6^{16}. A. *7-fold points.* The reference octahedron is one of a set of 135 octahedra all of which are conjugate under G. A particular octahedron is associated with a Göpel system of 7 mutually syzygetic half periods. No two of these octahedra have a vertex in common, whence the number of 7-fold points of M_6^{16} is $8 \cdot 135 = 1080$.

B. *4-fold* S_3*'s.* To each half-period of the theta functions there is associated an element J of period four whose square I is in G_{64}. As an involutorial element in G_{64}, I has a locus of fixed points made up of the two skew S_3's. Thus there is a correspondence between the 63 half-periods and 63 pairs of skew S_3's. A typical pair of S_3's is $z = z_1 = z_2 = z_6 = 0$, $z_3 = z_4 = z_5 = z_7 = 0$. From the form of (2), we have the following theorem.

THEOREM 2. *Each of the* 126 S_3*'s determined by the* 63 *half-periods lies on* M_6^{16} *and is a* 4*-fold locus of it.*

The sum of two half-periods is a third. Such a linearly related triad of half-periods will contain pairs which are either (a) syzygetic or (b) azygetic. We examine the intersections of the three pairs of S_3's determined by such syzygetic or azygetic triads. The number of such triads is 315 or 336 as the case may be.*

In case (a) as it appears for $p = 2$, the syzygetic triad of half-periods determines three pairs of lines which are the three pairs of opposite edges of a 4-point in S_3. For $p = 3$, however, they determine three pairs of S_3's which are the three pairs of opposite S_3's on a 4-line λ in S_7. Thus each line λ is on three S_3's. It is a line joining two vertices of one of the 135 reference octahedra and each line λ is on three pairs of such vertices. These 1260 lines λ are an extension of the 60 points of Klein's 60_{15} configuration ($p = 2$). We may state the following result.

THEOREM 3. *The* 126 S_3*'s determined by the* 63 *half-periods intersect by threes in* 1260 *lines* λ *which are* 6*-fold lines of* M_6^{16}. *These lines constitute one extension of Klein's* 60_{15} *configuration* ($p = 2$), *the other being the* 135 *reference octahedra. The* 30 *lines* λ *in each* S_3 *form the edges of a* 60_{15} *configuration* ($p = 2$).

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* Coble, Colloquium, loc. cit., §33.

Thus a plane through three vertices of a reference octahedron in S_7 cuts the manifold in an 18-ic, and therefore lies on the manifold. Any one of these planes is a 4-fold locus on M_6^{16}.

In case (b), and for $p=2$, the azygetic triad of linearly related half-periods determines three pairs of lines which are generators of one ruling of the quadric in S_3 which is determined by that even theta function which has the three half-periods as zeros. For $p=3$, they determine three pairs of S_3's any two of which are skew to each other. These six S_3's are S_3-generators of one ruling* for each of the quadrics in S_7 determined by each of the six even theta functions which has the three half-periods as zeros.

4. *Linear Sections of* M_6^{16}. We examine only those linear sections of M_6^{16} which are most effective for its projective determination. We have already noted that the S_3 of the type $z=z_1=z_2=z_6=0$ lies on M_6^{16}. On the other hand, from the equation of M_6^{16}, we have the following theorem.

THEOREM 4. *An* S_3 *of the type* $z=z_1=z_2=z_3=0$ *cuts* M_6^{16} *in the four faces of a tetrahedron each repeated four times.*

We shall also need the sections M_4^{16} of M_6^{16} by S_5's of the type $z=z_1=0$. Each of the 135 reference octahedra determines 28 of these S_5's but each S_5 occurs in three octahedra, whence there are $45\cdot 28$ of these S_5's and each is invariant under a subgroup $G_{3.48.2^9}$ of G. The factor 3 of this order is due to the three octahedra which contain the S_5; the factor 48 is the order of the subgroup of permutations of $z, \cdots, z_7$ which leaves the pair z, z_1 unaltered; and the factor 2^9 represents the multiplicative subgroup of $z, \cdots, z_7$ which appears in G.

Such an S_5 is cut by the 63 pairs of S_3's of Theorem 2 in pairs of linear spaces of the following character: (α) 3 consisting of an S_3 and an S_1; (β) 12 consisting of pairs of S_2's; (γ) 48 consisting of pairs of S_1's. According to Theorem 2 these are four-fold loci on M_4^{16}.

THEOREM 5. *An* M_4^{16} *cut out on* M_6^{16} *by an* S_5 *of type* $z=z_1=0$ *has the loci* (α), (β), (γ) *and the faces described in Theorem* 4 *as four-fold loci.*

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* Bertini, loc. cit., p. 143.

The S_1's of (α) are actually 8-fold loci on M_4^{16}. It is now relatively easy to prove the following theorem.

THEOREM 6. *M_4^{16} is the manifold of lowest order in S_5 invariant under $G_{48 \cdot 3 \cdot 2^9}$ with the multiplicities described in Theorem* 5.

For its invariance under the multiplicative group of order 2^9 rules out all terms whose exponents do not satisfy certain congruences. Its invariance under the permutation group of order 48 mentioned above necessitates the equality of various sets of coefficients. The remaining indeterminations are then easily removed by applying the multiplicity conditions of Theorems 4, 5.

5. *Determination of a Unique Manifold of Lowest Order with Multiplicities of* §3, *Invariant under the Group G of Order* $8! \cdot 36 \cdot 64$. The original form of the equation of M_6^{16} was obtained from transcendental considerations. But the invariance of M_6^{16} under the multiplicative group of order 2^9, $(J_{abc,lmn})$, is sufficient to determine the type of terms which may appear in its equation. This is accomplished by the solution of congruences. Its invariance under the permutation group $G_{8 \cdot 168}$ of $z, \cdots, z_7$ reduces the number of unknown coefficients to 8. These coefficients may now be determined by applying Theorem 6 to the S_5's defined by the reference octahedron. Thus a characteristic projective property of the manifold, M_6^{16}, originally defined by considerations in which function theory is essential, has been obtained. We may then make the following statement.

THEOREM 7. *M_6^{16} is the manifold of lowest order in S_7 invariant under the group G of order* $8! \cdot 36 \cdot 64$ *with the multiplicities of* §3.

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