NOTE ON THE LAW OF BIQUADRATIC RECIPROCITY*

BY V. W. ADKISSON

In outlining the proof of the law of biquadratic reciprocity H. J. S. Smith develops the expressions for S, T, S^4, and T^4† which are used in the proof given by Eisenstein.‡ We give here a slightly different development for these values making use of certain relationships established by Lebesgue.§ The advantage in this development lies in the fact that the function $\psi(i)$ is exhibited in a form which shows it to be a polynomial in i with integral coefficients, and that $\pm\psi(i)$ is a primary prime in the realm $k(i)$ if the proper sign be chosen. If

$$F(\alpha) = \sum_{n=0}^{p-2} \alpha^n x^{g^n},$$

where α is a root of the equation $(\alpha^{p-1}-1)/(\alpha-1)=0$, x is a root of $(x^p-1)/(x-1)=0$, g is a primitive root of p, and p is a prime of the form $4n+1$, then

$$F(\alpha)F(\alpha^{-1}) = \alpha^{(p-1)/2}p. \tag{1}$$

Substituting i for α, we obtain the result

$$F(i)F(i) = F(-1)\sum_{t=1}^{p-2} i^{\operatorname{ind} t}(-1)^{\operatorname{ind}(t+1)}.\| \tag{2}$$

Let

$$\psi(i) = \frac{[F(i)]^2}{F(-1)} = \sum_{t=1}^{p-2} i^{\operatorname{ind} t}(-1)^{\operatorname{ind}(t+1)}. \tag{3}$$

Hence, $\psi(i)$ is a polynomial in i with integral coefficients, and may be written in the form $a+bi$ where a and b are integers. But

---

* Research paper No. 280, University of Arkansas.

† H. J. S. Smith, *Collected Mathematical Papers*, vol. 1, pp. 78–87.

‡ E. Eisenstein, *Lois de réciprocité*, Journal für Mathematik (Crelle), vol. 28, pp. 57–67.

§ L. M. V.-A. Lebesgue, *Démonstration de quelques formules d'un mémoire de M. Jacobi*, Journal de Mathématiques (Liouville), vol. 19, pp. 289.

‖ Lebesgue, loc. cit.

$$(4) \qquad \psi(i)\psi(-i) = \frac{[F(i)F(-i)]^2}{[F(-1)]^2} = \frac{[i^{(p-1)/2}p]^2}{(-1)^{(p-1)/2}p} = p.$$

Therefore, $\psi(i)$ and $\psi(-i)$ are primes in $k(i)$. Since there are an even number of terms in (3) for which $\operatorname{ind} t$ is odd, and an odd number of terms for which $\operatorname{ind} t$ is even, $\pm\psi(i)$ and $\pm\psi(-i)$ will be primary primes, $a+bi$ and $a-bi$ respectively, in $k(i)$ if the proper signs be taken in each case. Moreover, $\psi(-i)$ [or $-\omega(-i)$] is primary if $\psi(i)$ [or $-\psi(i)$] is primary, and hence $(a+bi)(a-bi)=p$.

Since g is a primitive root of p, we have

$$g^{(p-1)/2} \equiv -1, \bmod p; \quad g^{(p-1)/4} \equiv i, \quad \bmod p_1;$$

and

$$g^{(p-1)/4} \equiv -i, \bmod p_2,$$

where $p_1p_2=p$. Then $(g/p_1)_4=i$.*

Let $g^s \equiv k \bmod p$; then since

$$F(i) = \sum_{s=0}^{p-2} i^s x^{g^s},$$

we find

$$F(i) = \sum_{s=0}^{p-2} \left(\frac{g^s}{p_1}\right)_4 x^{g^s} = \sum_{k=1}^{p-1} \left(\frac{k}{p_1}\right)_4 x^k = S.$$

From (3) and (4),

$$[F(i)]^4 = [F(-1)]^2[\psi(i)]^2 = p(a+bi)^2 = pp_1^2 = S^4.$$

In like manner

$$F(-i) = \sum_{k=1}^{p-1} \left(\frac{k}{p_1}\right)_4^3 x^k = T,$$

and

$$T^4 = p(a-bi)^2 = pp_2^2.$$

UNIVERSITY OF ARKANSAS

---

* The symbol $(g/p_1)_4$ due to H. J. S. Smith is the power of i which is congruent to $g^{(p-1)/4}$ mod p_1.