SOME THEOREMS ON PLANE CURVES

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In applying Abel's theorem to hyperelliptic integrals, we are interested in the intersections of certain curves with a curve C of the type $y^2=f(x)$, where $f(x)$ is a polynomial. The functions used in the following are all polynomials of degree indicated by their subscripts. If $f_n(x)\equiv f_k(x)f_{n-k}(x)$ we may without any loss of generality assume that $n\geqq k\geqq n/2$ and this assumption will be made throughout.

LEMMA. *If C is the curve $y^2=f_n(x)\equiv f_k(x)f_{n-k}(x)$, c_1 the curve $y=f_k(x)$ and c_2 the curve $y=f_{n-k}(x)$, then all the finite points of intersection of c_1 and c_2 are on C, and the curve S whose equation is $y=[f_k(x)+f_{n-k}(x)]/2$ is tangent to C at each of these k points.*

Suppose (α, β) is any one of the k points of intersection of c_1 and c_2; then $\beta=f_k(\alpha)$ and $\beta=f_{n-k}(\alpha)$ and therefore $\beta^2=f_k(\alpha)f_{n-k}(\alpha)=f_n(\alpha)$, that is (α,β) is on C. Obviously S passes through the k points of interesection of c_1 and c_2 and hence meets C in these k points. Eliminating y from the equations of S and C we get

$$\left[\frac{f_k(x)+f_{n-k}(x)}{2}\right]^2-f_k(x)f_{n-k}(x)\equiv\left[\frac{f_k(x)-f_{n-k}(x)}{2}\right]^2=0$$

as the equation giving the abscissas of the $2k$ points of intersection of S and C. Since the left hand side of this equation is a perfect square each abscissa is counted twice, and therefore since, in S, y is a one-valued function of x, S is tangent to C at each of these k points.

As an immediate consequence of this lemma we have the following result.

THEOREM 1. *If C is the curve $y^2=\phi_n(x)$, where $\phi_n(e_i)=0$, $(i=1,\cdots,n)$, and (α,β), $(\beta\neq 0)$, is a point on C, and c_1 is the curve of the form $y=\phi_k(x)$ determined by (α,β) and any k of the points $(e_i, 0)$, and c_2 is the curve of the form $y=\phi_{n-k}(x)$ determined by (α,β) and the remaining $n-k$ of the points $(e_i, 0)$, then c_1 and*

c_2 have all their k points of intersection on C, and the curve S whose equation is $y=[\phi_k(x)+\phi_{n-k}(x)]/2$ is tangent to C at each of these k points.*

Since $\phi_n(x)=\phi_k(x)\phi_{n-k}(x)$ for $n+1$ values of x, we have $\phi_n(x)\equiv\phi_k(x)\phi_{n-k}(x)$ and the theorem follows from the lemma.

That all curves S of the form $y=g_k(x)$ which are tangent to a curve C of the form $y^2=g_n(x)$ at each of k points can be obtained by this process, is a consequence of the following theorem.

THEOREM 2. *If (α_i,β_i), $(i=1,2,\cdots,k)$, are k points on the curve C whose equation is $y^2=g_n(x)$ such that there exists a curve S of the form $y=g_k(x)$ which is tangent to C at each of these k points, and if the curve c_1 whose equation is $y=h_k(x)$ meets C in the k points (α_i,β_i) and the point $(e_\lambda,0)$, where e_λ is any zero of $g_n(x)$, then $h_k(x)$ is a factor of $g_n(x)$.*

Since S is tangent to C at each of the k points, the equation $g_k^2(x)-g_n(x)=0$ has the roots $\alpha_1,\alpha_2,\cdots,\alpha_k$, each counted twice, and since c_1 meets S in the k points (α_i,β_i), the equation $g_k(x)-h_k(x)=0$ has the roots $\alpha_1,\alpha_2,\cdots,\alpha_k$.

We have therefore

$$[g_k(x)-h_k(x)]^2\equiv\mu[g_k^2(x)-g_n(x)],$$

and hence

$$[g_k(e_\lambda)-h_k(e_\lambda)]^2=\mu[g_k^2(e_\lambda)-g_n(e_\lambda)];$$

but $h_k(e_\lambda)=g_n(e_\lambda)=0$, hence $\mu=1$, and we have

$$g_k^2(x)-2g_k(x)h_k(x)+h_k^2(x)\equiv g_k^2(x)-g_n(x),$$

or

$$g_n(x)\equiv h_k(x)[2g_k(x)-h_k(x)].$$

If c_1 is the curve $y=a_0x^k+a_1x^{k-1}+\cdots+a_{k-1}x+a_k$ determined by the k points (α_i,β_i) and one of the n points $(e_i,0)$, the coefficient a_0 may be zero and the degree of the right hand side less than k. For suppose we choose a particular one, say e_1, of

* Only finite points of intersection are considered here. In certain special cases when n is even and $k=\frac{1}{2}n$, c_1 and c_2 may coincide or they may have less than k finite points of intersection. The lemma and Theorem 1 are still true for these cases when finite points of intersection are considered.

the zeros of $g_n(x)$ and find that the expression on the right is of degree k; then it will have as zeros k of the zeros of $g_n(x)$, say $e_1, e_2, \cdots, e_k$. Then the curve $y=b_0x^k+b_1x^{k-1}+\cdots+b_{k-1}x+b_k$ determined by the k points (α_i, β_i) and one of the remaining points $(e_i, 0)$, say $(e_{k+1}, 0)$, will have its right hand side of degree $n-k$ at most. For suppose the right hand side of degree $m>n-k$; then it will have as zeros m of the zeros of $g_n(x)$ and and hence at least one of the $e_1, e_2, \cdots, e_k$ and therefore $a_0x^k+a_1x^{k-1}+\cdots+a_k=b_0x^k+b_1x^{k-1}+\cdots+b_k$ for at least $k+1$ values. But since $b_0x^k+b_1x^{k-1}+\cdots+b_k$ has at least one zero which is not a zero of $a_0x^k+a_1x^{k-1}+\cdots+a_k$ this is impossible. It follows as a consequence of Theorem 1 that the degree of the right hand side is either k or $n-k$ depending on which zero of $g_n(x)$ is chosen for determining the curve c_1.

If in the above the degree of $h_k(x)$ is k, the degree of $2g_k(x)-h_k(x)$ will be $n-k$; if we denote the latter by $h_{n-k}(x)$, we shall have $g_k(x)\equiv[h_k(x)+h_{n-k}(x)]/2$. That is, the curve S is $y=[h_k(x)+h_{n-k}(x)]/2$, where the curve $y=h_k(x)$ is determined by some k of the points $(e_i, 0)$ and one of the points (α_i, β_i), and the curve $y=h_{n-k}(x)$ is determined by the remaining $n-k$ of the points $(e_i, 0)$ and the same one of the points (α_i, β_i).

Thus far it has not been necessary to say anything about the nature of the zeros $e_1, e_2, \cdots, e_n$. When these zeros are distinct we have the following theorem.

THEOREM 3. *The number of curves of the type $y=g_k(x)$ which are tangent to a curve C of the type $y^2=g_n(x)$ at any fixed point (α, β) and at $k-1$ other points, is C_k^n for $k>n/2$ and $\frac{1}{2}C_k^n$ for $k=n/2$, provided that the zeros of $g_n(x)$ are distinct.*

For by Theorem 1 we get a curve of this type corresponding to any k of the zeros of $g_n(x)$ and by Theorem 2 all curves of this type are obtained by this process. It must be shown, therefore, that when $k>n/2$ the same curve cannot be obtained from two different sets of k zeros of $g_n(x)$. Suppose $y=\phi_k(x)$ and $y=\psi_k(x)$ are both of degree k and cut out the same set of k points (α_i, β_i) on C; then $\phi_k(x)$ and $\psi_k(x)$ must have at least one zero in common and therefore $\phi_k(x)\equiv\psi_k(x)$. If n is even and $k=\frac{1}{2}n$, then each set of k such points is cut out by two and only two of these curves by Theorem 1.

From Theorem 1, the ordinary construction for drawing a tangent to a conic at a point P on it, when the axes and vertices are known, follows immediately.

The following example is a rather interesting illustration of Theorem 1. Let C be the curve

$$y^2 = f_6(x) \equiv -x^6 + 14x^4 - 49x^2 + 36.$$

The zeros of $f_6(x)$ are $1, -1, 2, -2, 3, -3$. Let the curve $c_1: y = f_3(x)$ be determined by $(0, 6)$ $(1, 0)$ $(-1, 0)(3, 0)$ and the curve $c_2: y = g_3(x)$ be determined by $(0, 6)$ $(2, 0)$ $(-2, 0)$ $(-3, 0)$; then we have

$$f_3(x) \equiv 2x^3 - 6x^2 - 2x + 6,$$
$$g_3(x) \equiv -\tfrac{1}{2}x^3 - \tfrac{3}{2}x^2 + 2x + 6.$$

These curves c_1 and c_2 meet on C in three points whose abscissas are 0, $(9+\sqrt{241})/10$, $(9-\sqrt{241})/10$. The curve S whose equation is

$$y = \frac{f_3(x) + g_3(x)}{2} \equiv \frac{3}{4}x^3 - \frac{15}{4}x^2 + 6$$

is tangent to C at each of these three points.

If we take for c_1 the curve $y = g_0(x)$ determined by $(0, 6)$, and for c_2 the curve $y = g_6(x)$ determined by $(0, 6)$ $(1, 0)$ $(-1, 0)$ $(2, 0)$ $(-2, 0)$ $(3, 0)$ $(-3, 0)$, we get

$$g_0(x) \equiv 6, \qquad g_6(x) \equiv -\frac{1}{6}x^6 + \frac{7}{3}x^4 - \frac{49}{6}x^2 + 6.$$

The curves c_1 and c_2 are each tangent to C at each of the three points $(0, 6)$ $(\sqrt{7}, 6)$ $(-\sqrt{7}, 6)$ and the curve S whose equation is

$$y = \frac{g_0(x) + g_6(x)}{2} = -\frac{1}{12}x^6 + \frac{7}{6}x^4 - \frac{49}{12}x^2 + 6$$

meets C four times at each of the three points.

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