A PROPERTY OF CONTINUA EQUIVALENT TO LOCAL CONNECTIVITY*

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1. *Introduction.* It is the purpose of this note to develop a property of continuous curves which seems to have been overlooked by the numerous writers on this subject and which, in effect, gives a new definition of this extensive class of continua. The property in question is given by the following definition.

A continuum (or space) M is said to be divisible if, for every pair of sub-continua A and B without common points, there is a decomposition of M into two continua P and Q such that $P \cdot B = Q \cdot A = 0$.

It will be shown that for a compact, metric, and connected space the concepts of divisibility and local connectivity at every point are equivalent. It will also be shown that this is true for any continuum, bounded or unbounded, located in a euclidean space.

As a preliminary we note that we cannot replace the word "sub-continua" in the above definition by the word "points." To see the truth of this statement consider the plane continuum M consisting of a segment ab of length 1 and an infinite set of arcs of radii n ($n=2, 3, \cdots$), each subtended by ab and of length less than a semi-circumference. It is readily seen that this continuum is not divisible in the sense of the above definition, but that for any two *points* A and B we can decompose M into two continua P and Q such that $P \cdot B = Q \cdot A = 0$.

2. THEOREM. *Let M be a metric, separable, connected space which is locally connected at each point. Then M is divisible.*

PROOF. Let A and B be any proper sub-continua of M and $A \cdot B = 0$. Let x be any point of A, let $\epsilon > 0$ be less than one-third the distance between x and B, and let the symbol $V_\epsilon(x)$ denote the set of points of M whose distance from x is less than

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ϵ. Since M is locally connected at each point, there is a $\delta > 0$ and a sub-continuum C_x of M such that $V_\delta(x) \subset C_x \subset V_\epsilon(x)$. Let F be the union of the continua $\{C_x\}$ as x ranges over A plus the limiting points of this set. Then F is a continuum, every point of A is an inner point of F, and $B \cdot F = 0$. Only the last statement requires explanation. If B contains a point a of F, it lies on no C_x, but is the limit of a sequence $\{a_n\}$ of points such that each a_n lies on some C_n corresponding to a point x_n of A. Let ϵ_n be the ϵ corresponding to x_n. These quantities can have no positive lower bound, since each ϵ_n is less than one-third the distance from x_n to B. But this means that for a partial sequence $\epsilon_n \to 0$, whence $x_n \to a$, contrary to the hypothesis that $A \cdot B = 0$. Thus $B \cdot F = 0$.

In like manner it is easy to see that there is a sub-continuum G of M such that every point of B is an inner point of G and $A \cdot G = 0$. Since M is everywhere locally connected, each component of $M - (F+G)$ has limiting points on $F+G$. Set P' equal to the union of F and all the components of $M-(F+G)$ having limiting points on F, and define Q' corresponding in like manner to G. Then $M = P' + Q'$. Also $P = \overline{P}'$ and $Q = \overline{Q}'$ are continua and, since A and B are inner sets of F and G, respectively, $P \cdot B = Q \cdot A = 0$. Thus M is divisible.

Note. Similar reasoning shows that, if $A_1, A_2, \cdots, A_n$ are proper sub-continua of M and $A_i \cdot A_j = 0$ if $i \neq j$, then M can be decomposed into n continua $\{P_i\}$ such that for each i, we have $A_i \subset P_i$ and $P_i \cdot [\sum_1^n A_i - A_i] = 0$.

3. THEOREM. *Let M be a compact, metric, and connected space which is divisible. Then M is locally connected at each point.*

PROOF. Let x be any point of M, let $\epsilon > 0$, and let K be any component of $M - V_\epsilon(x)$. Since M is divisible, $M = P_k + Q_k$, where P_k and Q_k are continua and $K \cdot P_k = x \cdot Q_k = 0$. Now $R_k = M - P_k$ is a region and $M - V_\epsilon(x)$ is contained in the union of the regions $\{R_k\}$.

Since M is compact, and $M - V_\epsilon(x)$ is closed, the Borel theorem applies and a finite set of the regions $\{R_k\}$, say R_1, $R_2, \cdots, R_n$, covers $M - V_\epsilon(x)$. As each R_i is a sub-set of a corresponding continuum Q_i which does not contain x, we have shown that $M - V_\epsilon(x)$ is contained in n sub-continua of M, none of which contains x. Since n is finite, it follows that for each $\epsilon>0$ there is a least integer m such that $M-V_\epsilon(x)\subset\sum_1^m S_i$, where each S_i is a continuum, $x\cdot S_i=0$, and $S_i\cdot S_j=0$ if $i\neq j$.

Let $R=M-\sum_1^m S_i$. Then R is a region and $R\subset V_\epsilon(x)$. Let ρ be any component of R not containing x and μ be the component containing x. No component ρ has limiting points on more than one S_i. For, if such a ρ existed, $x\cdot\bar\rho=0$ and $S_i+\bar\rho+S_j$ would be a continuum not containing x, contrary to the assumption that $M-V_\epsilon(x)$ is contained in not less than m such continua. On the other hand, each component of R has limiting points on at least one S_i. Since $\bar R$ is a compact closed set, no two continua $\{S_i\}$ have common points, and $M=\bar R+\sum_1^m S_i$ is a continuum, it follows that $\bar R$ contains a continuum H irreducible between S_1 and $\sum_2^m S_i$. Then $H'=H-H\cdot\sum_1^m S_i$ is connected and lies on a component of R. As $\bar H'$ contains points of more than one S_i, $H'\subset\mu$. Hence $\bar\mu\cdot S_1\neq 0$, and likewise $\bar\mu\cdot S_i\neq 0$ for every i.

If, for every $\epsilon>0$, there is some $\delta>0$ such that $\mu\supset V_\delta(x)$, the theorem is proved. For $\bar\mu$ is a continuum of diameter less than or equal to 2ϵ, which makes the oscillation of M about x not greater than 2ϵ. As ϵ can be chosen at pleasure, the oscillation is zero.

The alternative possibility is that for some $\epsilon>0$ and every $\delta>0$, $V_\delta(x)$ contains a point of some ρ. We show that this leads to a contradiction as follows. If this case occurs, every $V_\delta(x)$ must contain an infinite sequence of points $\{a_j\}$ converging to x, where each a_j lies on a component ρ_j of R and every $\bar\rho_j$ has points on precisely one S_i, say S_1. (This last statement is valid since m is finite and no component of R except μ has limiting points on more than one S_i.)

Since M is divisible, there is a decomposition $M=N_1+N_2$, where N_1 and N_2 are continua and $N_1\cdot S_1=N_2\cdot x=0$. Then N_1 contains an infinity of the points $\{a_j\}$, because x is a limiting point of the set $\{a_j\}$. Since $x\cdot N_2=0$ and $S_1\subset N_2$, we have $N_2\cdot S_i=0$ for $i\geqq 2$. For S_1 can be joined to no other S_i by a continuum not containing x.

Let L be a sub-continuum of N_1 irreducible between some a_j and $x+\sum_2^m S_i$, and let $L'=L-L\cdot(a_j+x+\sum_2^m S_i)$. Now L' is connected, $L'\subset M-\sum_1^m S_i$, and either $x\subset L'$ or $\bar L'\cdot S_i\neq 0$ for some $i\geqq 2$. In the first case $L'+x+a_j$ is connected and hence $a_j \subset \mu$, a contradiction. In the second case the connected set $L'+a_j \subset \rho_j$ and $\bar{\rho}_j \cdot S_i \neq 0$ for $i=1$ and some other i, another contradiction. Thus the assumption that there is no δ such that $V_\delta(x) \subset \mu$ is false and the theorem is proved.

4. THEOREM. *Let M be a bounded or unbounded continuum lying in a euclidean space. For M to be locally connected at every point it is necessary and sufficient that, for every pair of sub-continua A and B of M, there is a decomposition of M into two continua P and Q such that $P \cdot B = Q \cdot A = 0$.*

PROOF. The condition is necessary by §2. If M is bounded it is sufficient by §3. Let us assume then that M is unbounded. The proof for this case is obtained by suitably modifying that of §3 to take care of the fact that M is not compact.

Regarding M as a space, proceed as in the first paragraph of the proof of §3. Let F be the frontier of $V_\epsilon(x)$. As F is compact, it is covered by a finite set of the continua $\{Q_k\}$, say $Q_1, Q_2, \cdots, Q_n$. Then $T = M - V_\epsilon(x) + \sum_1^n Q_i$ is a closed set and a finite number of its components cover F. Let these be $C_1, C_2, \cdots, C_k$. It is easily seen that $T = \sum_1^k T_i$, where each T_i is a closed set, $T_i \cdot T_j = 0$ if $i \neq j$, and $C_i \subset T_i$ for each i. If T_1 is not a continuum, $T_1 = U_1 + W_1$, where $U_1 \cdot W_1 = 0$ and U_1 and W_1 are closed; suppose that $C_1 \subset U_1$. Then M is the sum of two closed sets, W_1 and $V_\epsilon(x) + U_1 + \sum_2^m T_i$, without common points, —an impossibility as M is a continuum. Thus $C_1 = T_1$ and likewise $C_i = T_i$ for every i. We have then, as in the second paragraph of the proof of §3, for each ϵ a least integer m such that $M - V_\epsilon(x)$ is covered by m mutually exclusive sub-continua of M, which we may denote by $\{S_i\}$ and none of which contains x.

The remainder of the proof of §3 needs no alteration for this case. The existence of the irreducible continuum L in the last paragraph can be established in the same way as the existence of H in the third paragraph.

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