$$t = \left[\frac{\log n - \log 2}{\log 3}\right].$$

Therefore

$$R(n) \geqq \left[\frac{\log n - \log 2}{\log 3}\right] + 1.$$

Hence

$$\left[\frac{\log n}{\log 2}\right] \geqq R(n) - 1 \geqq \left[\frac{\log \frac{n}{2}}{\log 3}\right].$$

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MULTIPLE POINTS OF ALGEBRAIC CURVES*

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1. *Introduction.* Limits to the number of multiple points of algebraic curves were first found by Cramer.† He found and tabulated the maximum numbers of multiple points of all possible orders for curves of orders up to and including eight. Plücker‡ obtained the general expression $(n-1)(n-2)/2$ for the maximum number of double points of an algebraic curve of order n.

Except for individual curves, the maximum number of multiple points of higher order than two for a curve of given order has not been found. A general expression for the maximum number of compound singularities or singularities of different orders is not practicable. When, however, the curve possesses only multiple points or sets of multiple points of the same order, serviceable limits for the maximum number of such singularities can be found.

The purpose of this paper is to determine the maximum number of distinct multiple points of given order and con-

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† G. Cramer, *Introduction à l'analyse des lignes courbes algébriques*, Geneva, 1750, pp. 455–459.

‡ J. Plücker, *Theorie der Algebraischen Curven*, Bonn, 1839, p. 215.

secutive multiple points of given order that may belong to an algebraic curve of given order.

2. *Invariants Associated with Multiple Points.* With each singularity of a curve f there are associated certain functions of the coefficients of the equation of f, invariant under a projective transformation, whose vanishing is necessary and sufficient for f to possess this singularity. Such functions will be called invariants. Also, all curves herein considered are proper curves.

A necessary and sufficient condition for f to have a multiple point of order r at P is that all the partial derivatives of f up to and including the $(r-1)$st vanish at P. These partial derivatives constitute $r(r+1)/2$ linear functions of the coefficients of f. The postulation of an ordinary r-fold point on f is, therefore, $r(r+1)/2$. The two independent coordinates of P may be eliminated from the $r(r+1)/2$ relations among the coefficients in $r(r+1)/2-2$ independent ways giving rise to $r(r+1)/2-2$ invariants associated with each ordinary r-fold point.

Consecutive multiple points of f are multiple points that coincide at a point P in such a way that all the branches of f common to two or more of these multiple points have a common tangent at P. Enriques* has proved that the postulation of s consecutive multiple points of orders r_i on f is the same as the total postulation of the s multiple points considered as distinct on f, that is

$$\frac{1}{2}\sum_{i=1}^{s} r_i(r_i+1).$$

Assume that s points are consecutive on a curve. If one of the points P is fixed, one condition determines the direction of approach to P of each of the remaining $s-1$ points. The two parameters defining P and the $s-1$ parameters determining the $s-1$ directions through P total $s+1$ parameters.

* F. Enriques, *Lezioni sulla Teoria Geometrica delle Equazione e delle Funzioni Algebriche*, vol. 2, pp. 404–408.

Therefore the number of independent parameters involved in the location of s consecutive points on any plane curve is $s+1$.

The postulation of a singularity is the total number of conditions necessary and sufficient to determine both the nature of the singularity and its position. The $\sum r_i(r_i+1)/2$ relations among the coefficients of f then involve the $s+1$ parameters which determine the positions of the s consecutive points. From these $\sum r_i(r_i+1)/2$ relations the $s+1$ parameters may be eliminated in $\sum r_i(r_i+1)/2-s-1$ independent ways, each eliminant being an invariant associated with the singularity. There are, then,

$$\frac{1}{2}\sum_{i=1}^{s} r_i(r_i+1) - s - 1$$

invariants associated with s consecutive multiple points of orders r_i. These invariants determine the nature of this singularity of f and only that.

3. *The Maximum Number of Consecutive Multiple Points.* It has been proved by Lefschetz* that there can be no more that $n(n+3)/2-8$ invariants among the coefficients of the equation of an algebraic curve of order n and genus $p>1$. In the preceding section it has been found that $sr(r+1)/2-s-1$ invariants are associated with s consecutive r-fold points. A limit to the number of consecutive r-fold points that may occur on a curve of order n and genus $p>1$ is therefore determined by the inequality

$$\tfrac{1}{2}sr(r+1) - s - 1 \leqq \tfrac{1}{2}n(n+3) - 8$$

which, when solved for s, becomes

$$s \leqq \frac{n^2+3n-14}{(r-1)(r+2)}. \tag{1}$$

* S. Lefschetz, *On the existence of loci with given singularities*, Transactions of this Society, vol. 14 (1913), pp. 23–24.

For $r=2$, this is the limit to the number of consecutive nodes of a curve of order n found by Sharpe and Craig.*

Another limit to the number of consecutive r-fold points of a curve of order n is found from the fact that the total number of double points cannot exceed $(n-1)(n-2)/2$. Since an r-fold point contains $r(r-1)/2$ nodes, this gives the limit

$$s \leqq \frac{(n-1)(n-2)}{r(r-1)}. \tag{2}$$

For $r=2, 3, 4$, limit (1) holds for $n \geqq 6, 10, 13$, respectively, otherwise limit (2). Both limits give $s=10$ for $r=2$ and $n=6$. For $n \leqq 6$, therefore, all the nodes of an algebraic curve of any genus may be consecutive.

Limits (1) and (2) are equal for

$$r = \frac{(n-1)(n-2)}{3n-8}.$$

Limit (1) holds for all values of r less than or equal to the above value and limit (2) holds for all values of r greater than or equal to this value. However, there are certain exceptions to limit (2) now to be discussed.

Since limit (2) is determined by the maximum number of double points of the curve, this limit would permit a curve of order n to have more than one point of order $r \geqq (n+1)/2$. This is impossible, since a line joining two such points would intersect the curve in more than n points. The range of r for limit (2) is, therefore, restricted to $(n-1)(n-2)/(3n-8) \leqq r \leqq n/2$.

Within the above range for r, limit (2) holds except when the value of s determined by limit (2) is of the form $n'(n'+3)/2$, where n' is any positive integer. A curve of order n' through the s consecutive r-fold points of f cannot intersect f in more than nn' points. In case the value of s is not the exact number necessary to determine a curve of order

* Sharpe and Craig, *Plane curves with consecutive double points*, Annals of Mathematics, vol. 16 (1914–15), p. 19.

n', then additional simple points of f must be chosen to determine the curve of order n' through all the s consecutive r-fold points of f. In this case, the number of intersections of the two curves is always less than nn' when limit (2) is satisfied.

There remains, then, an exception to limit (2) for $r \leqq n/2$ only when limit (2) gives the value $s = n'(n'+3)/2$ and then only when $nn' < sr$. We shall now prove that in all cases when $n > 4$, $s = n'(n'+3)/2$ and $nn' < sr$, the inequality $sr - nn' < r$ holds.

Assume $s = n'(n'+3)/2 = (n-1)(n-2)/[r(r-1)]$ and eliminate n' from this equation and the inequality $sr - nn' < r$ to be proved. This gives

$$3nr^2 - r(n^2+6n-2) - (n-1)(n-2) < 0.$$

This inequality is evidently satisfied except for large values of r. The largest value of r occurs when $s = 5$. If we eliminate r from the equality $5r(r-1) = (n-1)(n-2)$ and the above inequality, there results an inequality in n which is satisfied for all values of $n > 4$.

Therefore, when the value of s defined by limit (2) is of the form $n'(n'+3)/2$ and $nn' < sr$, since also $sr - nn' < r$, the maximum number of consecutive r-fold points for that order n is one less than the value given by limit (2).

The least value of r for which limit (2) holds is $(n-1)(n-2)/(3n-8)$. Substitute this value of r in limit (2) and there results $s = 9$ for $n > 6$. Substituting the largest value $n/2$ of r in limit (2), we obtain $s = 3$. The entire range of values of s given by limit (2) for $n > 6$ is, therefore, $9 \geqq s \geqq 3$.

When $s = 9$, a cubic can be passed through the nine multiple points, that is, $n' = 3$. Since $(n-1)\ (n-2)/(3n-8) - n/3$ is less than $1/3$ for all values of $n > 6$, the largest value of r in proportion to n for which $s = 9$ occurs when n is a multiple of 3 and $r = n/3$. In this case, a cubic through the nine multiple points intersects f in $3n$ points, that is, all the intersections of f and the cubic occur at the multiple points. The only restriction on the positions of the nine multiple points of order $n/3$ is that they can not all lie at the nine intersections of two cubics.

The only case in which $nn' < sr$, therefore, occurs when $n' = 2$ and $s = 5$. If we set $s = 5$ in limit (2), there results a value of r in terms of n such that five times this value of r is always greater than $2n$. Then, in accordance with the above proof, when $r > 2n/5$ and limit (2) gives $s = 5$, the maximum number of consecutive multiple points of this multiplicity is four. This is the only exception to limit (2) within the range of the values of r for which it holds.

Since limit (2) is defined by the maximum number of double points of a curve, this limit as well as the above discussion of it applies equally well to distinct multiple points and to consecutive multiple points.

When r has its maximum value for a given n, if n is even, $r = n/2$, $s = 3$ and if n is odd, $r = (n-1)/2$, $s = 4$ for $n > 5$. These multiple points may be either distinct or consecutive.

If k cusps, $0 \leqq k \leqq r-1$, replace the same number of nodes in s consecutive r-fold points, the limit for s obtained by considering the number of invariants associated with the singularity is

$$s \leqq \frac{n^2 + 3n - 14 - 2k}{(r-1)(r+2)} \cdot \tag{3}$$

Limit (2) is, however, unchanged. For given values of r, k and n, the smaller of the two limits (2) and (3) now determines the maximum value of s. In case the inequality $k \geqq 3n - 8 - (n-1)\ (n-2)/r$ is satisfied, limit (3) holds, otherwise limit (2). The same restrictions on limit (2) obtained above for $k = 0$, hold for all values of k since in limit (2) and in determining the number of intersections of two curves, a cusp counts merely as a double point.

If a curve f of order n has j singularities each consisting of s consecutive nodes, the following limits are found:

(a) $$j \leqq (n^2 + 3n - 16)/(4s - 2),$$

(b) $$j \leqq (n-1)(n-2)/(2s).$$

For $s \leqq (n-1)(n-2)/[(n-4)(n-5)]$ limit (a) holds, otherwise limit (b).

For $s=2$, solve $(n-1)(n-2)/[(n-4)(n-5)] \geqq 2$ for n and we obtain $n \leqq 11$. For $n=12$, however, 54 of its possible 55 nodes may form 27 tacnodes and limit (a) still be satisfied. Then for $n \leqq 12$, a curve of any genus may have all its nodes consecutive in pairs to form tacnodes when the number of nodes is even or all but one when the number is odd. Similarly, for $n \leqq 9$ a curve of any genus may have as many oscnodes as the integral number of times three is contained in the number of its nodes.

If each of the singularities consists of $s-1$ nodes and one cusp, all consecutive, each singularity accounts for $2s$ invariants. Then the maximum number j of such singularities must satisfy limit (b) and also the limit

$$\text{(a')} \qquad j \leqq (n^2 + 3n - 16)/(4s).$$

Limit (a′) is less than limit (b) for $n \geqq 6$ and all values of s. For example, a quintic may have three ramphoid cusps or two tacnode-cusps. A sextic may have three tacnode-cusps, but not five ramphoid cusps nor two sets each containing four nodes and one cusp.

More generally, in order that a curve of order n have j singularities each consisting of s consecutive r-fold points, the following inequality must be satisfied:

$$j \leqq (n^2 + 3n - 16)/[s(r-1)(r+2) - 2],$$

and also the total number of nodes in the j singularities must not exceed $(n-1)(n-2)/2$.

In order to show that a curve of order n can possess a singularity consisting of s consecutive multiple points of orders $r_1, r_2, \cdots, r_s$, both of the following inequalities must be satisfied:

$$\sum r_i(r_i + 1) - 2s \leqq n(n+3) - 14,$$
$$\sum r_i(r_i - 1) \leqq (n-1)(n-2),$$

and also it must be shown that the sum of the $n'(n'+3)/2$

largest values of r_i does not exceed nn' for any value of n' such that $s \geqq n'(n'+3)/2$.

None of the above limits hold when more than two multiple points are collinearly consecutive. Since a line can not intersect a curve of order n in more than n points, the limit to the number of collinearly consecutive r-fold points is $s \leqq n/r$, or in case the multiple points are of different orders, the inequality $\sum r_i \leqq n$ must be satisfied.

4. *Limits for Distinct Multiple Points.* Since each r-fold point involves $r(r+1)/2-2$ invariants, the following limit defines the maximum number s of distinct r-fold points:

$$(1') \qquad s \leqq (n^2 + 3n - 16)/(r^2 + r - 4).$$

It was noted in the preceding section that limit (2) applies to distinct as well as to consecutive multiple points. Then limits (1′) and (2) apply to distinct multiple points. The discussion of limit (2) in the preceding section also applies equally well to distinct multiple points.

Limit (1′) holds for

$$r \leqq \{n^2 - 7 + [(n^2 - 7)^2 - 24(n - 1)(n - 2)(n - 3)]^{1/2}\}/[6(n - 3)],$$

otherwise limit (2). The radicand is negative for $n \leqq 16$, but for a given value of r the least value of n for which limit (1′) is less than limit (2) is 18.

For $r=2$, limit (1′) is always greater than limit (2). For $r=3$ or 4, limit (2) holds for $n \leqq 17$ and limit (1′) for $n \geqq 18$. As r increases, this discriminating value of n increases, for example, when $r=5$ or 6, limit (1′) holds for $n \geqq 20$, 23 respectively, otherwise limit (2).

We shall now prove that neither limit (1) for the maximum number of consecutive multiple points of order r nor limit (1′) for the maximum number of distinct multiple points of order r is subject to any restriction due to intersections of f with a curve through its multiple points.

For $r[(4n^2+12n-47)^{1/2}-1]/2$, limit $(1') \geqq$ limit (1). For

$n \geqq 4$, this value of r is greater than the upper limiting value of r for which limit (1) holds. Then limit (1′) is larger than limit (1) for all values of n and r for which limit (1) holds. Then to prove that limit (1′) is never so large as to allow a curve of order n' through the s multiple points to have more than nn' intersections with the curve of order n possessing the multiple points will also establish the same property for limit (1).

Let $s=(n^2+3n-16)/(r^2+r-4)=n'(n'+3)/2$, that is, assume the largest possible value of s given by limit (1′) and assume further that this is the exact number of points necessary to determine a curve of order n' through the s distinct multiple points of order r. These are the conditions under which a curve of order n' through the s multiple points intersects the curve possessing the multiple points in the greatest number of points.

Together with the above equality, assume $nn' \geqq sr$. Eliminate n' from this inequality and the above equality and there results

$$r^2(n^2 + 3n - 16) - n(2n - 3r)(r^2 + r - 4) \leqq 0.$$

Substitute in this inequality the maximum value of r in terms of n for which formula (1′) holds and the resulting inequality in n is satisfied by all values of $n \geqq 5$.

Then the maximum number of multiple points of order r as defined by limit (1′) when distinct, or by limit (1) when consecutive, satisfies the criterion that no proper curve of order n' can be described through them intersecting the curve of order n possessing the multiple points in more than nn' points.

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