the same manner, we obtain as the image of V_k^μ a $V_k^{\mu^2}$ which is of the same nature as that obtained by means of the r^2-ic transformation as the image of an S_{r-1}.

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ON SOME FUNCTIONS CONNECTED WITH $\phi(n)$

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Let $\phi(n)$ denote, as usual, the number of numbers not greater than and prime to n. Let $N(x)$ be the number of distinct numbers less than x, which can be the ϕ function of some number; and let $R(n)$ be the number of solutions of the equation

$$n = \phi(x),$$

n being given. The object of this note is to prove some results concerning the magnitude of $N(n)$ and to apply them to prove that

$$\overline{\lim_{n=\infty}}\, R(n) = \infty\ .$$

Since there is no reference to such results in Dickson's *History of the Theory of Numbers*, I believe that the last result in particular is new.

THEOREM I. *We have*

$$N(n) > \frac{a \cdot n}{\log n},$$

where a is a constant.

PROOF. For each prime p, $\phi(p) = p - 1$; hence, if we denote by $\pi(n)$ the number of primes not exceeding n, then

$$N(n) \geqq \pi(n).$$

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But by elementary methods it has been proved that*

$$\pi(n) > \frac{a \cdot n}{\log n}$$

where $a > 0$ is a constant. Therefore

$$N(n) > \frac{a \cdot n}{\log n} \cdot$$

THEOREM II. *We have*

$$N(n) = O\left\{\frac{n}{(\log n)^t}\right\},$$

where $t = (\log 2)/e$.

PROOF. If p is an odd prime, $\phi(p^\alpha)$ is even; and $\phi(m \cdot n) = \phi(m)\phi(n)$ when m and n are prime to each other. Therefore, if any number m is composed of more than r different odd prime factors, then $\phi(m)$ is divisible by 2^{r+1}. So, if a number of the form $2^r \cdot h$ (where h is odd) should be a $\phi(m)$, then m may contain at most r different odd prime factors. Consequently, in the set $2^s \cdot h_s$, where s takes the values $0, 1, 2, \cdots, r$, and h_s runs through all odd numbers not exceeding $n/2^s$, the number of numbers which can be the ϕ of some numbers, is not greater than

$$\pi_1(n) + \pi_2(n) + \cdots + \pi_{r+1}(n),$$

where $\pi_r(x)$ is the number of numbers not exceeding x, which are composed of r different prime factors. But the number of numbers in the set $2^s \cdot h_s$ considered above, is

$$= \sum_{0 \leqq s \leqq r} \left[\frac{n}{2^{s+1}}\right] = \sum_{0 \leqq s \leqq r} \frac{n}{2^{s+1}} + O(r) = n - \frac{n}{2^{r+1}} + O(r).$$

Hence, of the numbers $\leqq n$, at least

$$n - \frac{n}{2^{r+1}} + O(r) - \sum_{1 \leqq s \leqq r+1} \pi_s(n)$$

numbers cannot be the ϕ function of any number. Therefore

* Ramanujan's *Collected Papers*, pp. 208–209. Landau, *Vorlesungen über Zahlentheorie*, vol. 1; Theorem 112. etc.

$$N(n) \leqq n - \left\{ n - \frac{n}{2^{r+1}} + O(r) - \sum_{1 \leqq s \leqq r+1} \pi_s(n) \right\}$$

$$= \frac{n}{2^{r+1}} + O(r) + \sum_{1 \leqq s \leqq r+1} \pi_s(n).$$

By elementary methods, Hardy and Ramanujan have proved that*

$$\pi^{\gamma}(n) < \frac{kn(\log\log n + c)^{\gamma-1}}{(y-1)!\log n},$$

where k and c are constants.

Hence, if $r+1 < \log\log n$,

$$\sum_{1 \leqq s \leqq r+1} \pi_s(n) = O\left(\sum_{1 \leqq s \leqq r+1} \frac{n(\log\log n + c)^{s-1}}{(s-1)!\log n} \right)$$

$$= O\left(\sum_{1 \leqq s \leqq r+1} \frac{n(\log\log n)^{s-1}}{(s-1)!\log n} \right),$$

for

$$\left(1 + \frac{c}{\log\log n}\right)^{s-1} \leqq \left(1 + \frac{c}{\log\log n}\right)^{\log\log n}$$

$$= e^{\log\log n \log(1+c/\log\log n)} \leqq e^c = O(1).$$

But, since $r+1 < \log\log n$,

$$\frac{(\log\log n)^{s-1}}{(s-1)!} < \frac{(\log\log n)^s}{s!}.$$

Therefore, if $r+1 < \log\log n$,

$$\sum_{1 \leqq s \leqq r+1} \pi_s(n) = O\left(\frac{r \cdot n}{\log n} \frac{(\log\log n)^r}{r!} \right).$$

Therefore, if $r+1 < \log\log n$,

$$N(n) = O\left(\frac{n}{2^{r+1}}\right) + O(r) + O\left(\frac{n}{\log n} \frac{(\log\log n)^r}{(r-1)!} \right)$$

$$= O(s_1) + O(s_2) + O(s_3), \text{ say.}$$

Put

* Ramanujan's *Collected Papers*. Paper No. 35, 2.2, Lemma A.

$$r = \left[\frac{\log\ \log\ n}{e}\right].$$

Then, by Stirling's theorem,

$$\begin{aligned}
\log s_3 &= \log n - \log\log n + r\log\log\log n - (r-1)\log(r-1) \\
&\quad - \frac{1}{2}\log(r-1) + r - 1 + O(1) \\
&\leqq \log\log n - \log\log n + \frac{\log\log n\log\log\log n}{e} \\
&\quad - \frac{\log\log n}{e}\log\frac{\log\log n}{e} - \frac{1}{2}\log\frac{\log\log n}{e} \\
&\quad + \frac{\log\log n}{e} + O(1) \\
&= \log n - \log\log n\left(1 - \frac{1}{e} - \frac{1}{e}\right) - \frac{1}{2}\log\log\log n \\
&\quad + O(1) \leqq \log n - \log\log n\left(\frac{e-2}{e}\right) + O(1) \\
&\leqq \log n - \frac{\log 2}{e}\log\log n + O(1) \\
&= \log n - t\log\log n + O(1).
\end{aligned}$$

Therefore

$$s_3 = O\left(\frac{n}{(\log n)^t}\right),$$

$$s_2 = O(r) = O(\log\log n) = O\left(\frac{n}{\log^t n}\right),$$

$$s_1 = O\left(\frac{n}{2^{\log\log n/e}}\right) = O\left(\frac{n}{\log^t n}\right).$$

Therefore

$$\begin{aligned}
N(n) &= O(s_1 + s_2 + s_3) \\
&= O\left(\frac{n}{\log^t n}\right).
\end{aligned}$$

Now we shall apply the above result to prove the following theorem.

THEOREM III. *We have*

$$R(n) \neq o(\log^t n)$$

and in particular,

$$\overline{\lim_{n=\infty}}\, R(n) = \infty .$$

PROOF. Let

$$s(n) = \sum_{1 \leqq m \leqq n} R(n) .$$

If possible, let

$$R(m) = o(\log m)^t.$$

Then

$$\begin{aligned} s(n) &\leqq N(n)(\max_{1 \leqq m \leqq n} R(m)) \\ &= N(n)\left\{o(\log^t n)\right\} \\ &= o\left(\frac{n}{\log^t n}\right)\left\{O(\log^t n)\right\} \\ &= o(n) , \end{aligned}$$

from Theorem II.

Now, $S(n)$ is the number of numbers whose ϕ functions are $\leqq n$. But, since $\phi(m) < m$, $S(n) \geqq n$, which contradicts $S(n) = o(n)$. The theorem is therefore proved.

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