A CERTAIN BIRATIONAL TRANSFORMATION OF ORDER r^2 BETWEEN TWO r-SPACES IN AN $(r+1)$-SPACE

BY B. C. WONG

In this paper we present a birational transformation of order r^2 between two r-spaces S_r and S_r' in an $(r+1)$-space S_{r+1}. If $r=2$, we have the well known quartic birational transformation between two planes of an S_3 obtained by intersecting with the two planes the bisecants of a twisted cubic curve in S_3. A point of one plane and a point of the other are said to be corresponding points if they lie on the same bisecant of the cubic curve. It is our object to generalize this construction to hyperspace.

For this purpose we let two r-ic hypersurfaces (necessarily ruled) intersect in a composite manifold composed of an M_{r-1}^n of order $n=r(r+1)/2$ and an $M_{r-1}^{n'}$ of order $n'=r(r-1)/2$. The former component manifold, M_{r-1}^n, will have just one apparent r-fold point* if it meets the latter, M_{r-1}, in an $(r-2)$-dimensional variety of order $r(r^2-1)/3$. This implies that M_{r-1}^n and $M_{r-1}^{n'}$ are such that a general S_3 meets them in two curves C^n and $C^{n'}$, respectively, having $r(r^2-1)/3$ points in common and that C^n has $r(r^2-1)\cdot(3r-2)/24$ and $C^{n'}$ has $r(r-1)(r-2)(3r-5)/24$ apparent double points. It also implies that a general S_4 meets them in two surfaces having respectively $r(r+1)(r-1)^2(r-2)^2/48$ and $r(r-1)(r-2)^2(r-3)^2/48$ apparent triple points. The number of apparent $(k-1)$-fold points on their intersections with an S_k can also be found.†

As we are going to make use of M_{r-1}^n‡ to obtain an r^2-ic

* By an apparent r-fold point of a V_{r-1} in S_{r-1} we mean a line passing through a general point of S_{r+1} and meeting V_{r-1} r times.

† See B. C. Wong, *On the number of apparent triple points of surfaces in space of four dimensions*, this Bulletin, vol. 35, No. 3, pp. 339–343.

‡ There are in S_{r+1} many $(r-1)$-dimensional varieties of order m $[2r-1\leqq m\leqq r(r+1)/2]$ that have just one apparent r-fold point but the M_{r-1}^n here described is the only one that offers the desired transformation.

birational transformation between S_r and S_r', we make a few remarks concerning the manifold. It can be shown that this manifold can be represented upon an S_{r-1} by means of an ∞^{r+1}-system of $(r+1)$-ic $(r-2)$-dimensional varieties passing through an $(r-3)$-dimensional variety of order $(r+1)(r+2)/2$ in S_{r-1}. The case $r=3$ is well known: the points of M_2^6 are in a one-to-one correspondence with the points of a plane, the fundamental curves of representation being quartic curves through 10 points in the plane.

M_{r-1}^n is also the locus of points whose polar r-spaces with respect to r hyperquadrics $Q_r^{(i)}$ $[i=1, 2, \cdots, r]$ in S_{r+1} meet in lines all lying in an S_r. The polar r-spaces of points of S_r with respect to $Q_r^{(i)}$ meet in lines r-uply secant to M_{r-1}^n. Again, an S_r meets M_{r-1}^n in a V_{r-2}^n which is the Jacobian variety of the r quadric varieties in which S_r meets $Q_r^{(i)}$. The polar $(r-1)$-spaces of the points of V_{r-2}^n with respect to the same quadric varieties of S_r meet in lines intersecting V_{r-2}^n r times and forming a $V_{r-1}^{r^2-1}$ of order r^2-1 on which V_{r-2}^n lies r-uply.

If we replace r by $r-1$ in n, the M_{r-1}^n of S_{r+1} becomes an $M_{r-2}^{n'}$ of S_r which is of the same nature as the variety in which S_r meets $M_{r-1}^{n'}$. Now $M_{r-1}^{n'}$ is such that a general S_{r-1}-section of it is the Jacobian variety of $r-1$ $(r-2)$-dimensional quadric varieties in S_{r-1}. It is to be noticed that $M_{r-1}^{n'}$ is a ruled manifold composed of ∞^{r-2} lines all meeting M_{r-1}^n r times. These lines or the points of the variety in which an S_r meets $M_{r-1}^{n'}$ can be set in a one-to-one correspondence with the points of an S_{r-2}.

Now let two r-spaces S_r and S_r' be given in S_{r+1} and let S_r meet M_{r-1}^n in a V_{r-2}^n and S_r' meet M_{r-1}^n in a $V_{r-2}'^n$. These two varieties, V_{r-2}^n and $V_{r-2}'^n$, have in common a V_{r-3}^n which lies in the R_{r-1} of intersection of S_r and S_r'. The r-fold secants of V_{r-2}^n form a $V_{r-1}^{r^2-1}$ and those of $V_{r-2}'^n$ form a $V_{r-1}'^{r^2-1}$. These two r-fold secant varieties both intersect R_{r-1} in the same $V_{r-2}^{r^2-1}$ which contains V_{r-3}^n r-uply.

Since M_{r-1}^n has just one apparent r-fold point, from a point P of S_r we draw the r-fold secant to M_{r-1}^n meeting S_r' in a point P' which is said to correspond to P. As the construction is reversible, the correspondence is birational. To show that the correspondence is of order r^2, that is, to an $(r-1)$-space of one of the given r-spaces corresponds a $V_{r-1}^{r^2}$ of order r^2 of the other, we notice that the ∞^{r-1} r-uple secant lines of M_{r-1}^n that meet an S_{r-1} form a $V_r^{r^2}$, for an S_r containing S_{r-1} meets it in a composite variety composed of S_{r-1} and a $V_{r-1}^{r^2-2}$. Hence if P describes an S_{r-1} in S_r, the corresponding r-fold secant of M_{r-1}^n describes a $V_r^{r^2}$ which is met by S_r' in a $V'^{r^2}_{r-1}$. Similarly, there is a $V_{r-1}^{r^2}$ in S_r corresponding to an S_{r-1}' in S_r'.

Now $V_{r-1}^{r^2}$ passes through the variety V_{r-2}^n of intersection of M_{r-1}^n with S_r, r times; and through the $V_{r-2}^{r^2-1}$ in R_{r-1}, once. Since, from the very nature of the transformation, every point of R_{r-1} not on $V_{r-2}^{r^2-2}$ is its own image, $V_{r-1}^{r^2}$ has in common with R_{r-1}, besides the $V_{r-2}^{r^2}$, an S_{r-2} in which R_{r-1} meets the S_{r-1}' to which $V_{r-1}^{r^2}$ corresponds.

Now to show that to a line in one r-space corresponds an r^2-ic curve in the other. Let P describe a line l in S_r. The line l meets the variety $V_{r-1}^{r^2-1}$ of lines r-uply secant to the V_{r-2}^n in which S_r meets M_{r-1}^n in r^2-1 points, i.e., meets r^2-1 of the lines of $V_{r-1}^{r^2-1}$. Hence the ∞^1 r-fold secants of M_{r-1}^n that meet l form a ruled surface F^{r^2} of order r^2, for it is met by S_r in a composite curve made up of r^2 lines one of which is l. Now F^{r^2} is met by S_r' in a curve C'^{r^2} which is the image of l in S_r.

Similarly, it can be shown that, if a point describes a k-space in the one r-space, the corresponding point in the other r-space describes a k-dimensional variety of order $\binom{r}{k}^2$.

If we project S_r' upon S_r, we have an involutorial transformation of order r^2 in S_r. Attention is here called to the fact that this involutorial correspondence is the product of two involutorial r-ic correspondences. One such can be set up by means of r of the $r(r-1)$-dimensional quadric varieties $Q_{r-1}^{(i)}$ in S_r. To a point P we make correspond the point P' of intersection of the r polar $(r-1)$-spaces with respect to $Q_{r-1}^{(i)}$. If P describes an S_k, P' describes a V_k^μ, where $\mu=\binom{r}{k}$. By setting up in S_r another involutorial r-ic transformation in the same manner, we obtain as the image of V_k^μ a $V_k^{\mu^2}$ which is of the same nature as that obtained by means of the r^2-ic transformation as the image of an S_{r-1}.

THE UNIVERSITY OF CALIFORNIA

ON SOME FUNCTIONS CONNECTED WITH $\phi(n)$

BY S. SIVASANKARANARAYANA PILLAI*

Let $\phi(n)$ denote, as usual, the number of numbers not greater than and prime to n. Let $N(x)$ be the number of distinct numbers less than x, which can be the ϕ function of some number; and let $R(n)$ be the number of solutions of the equation

$$n = \phi(x),$$

n being given. The object of this note is to prove some results concerning the magnitude of $N(n)$ and to apply them to prove that

$$\overline{\lim_{n=\infty}} R(n) = \infty .$$

Since there is no reference to such results in Dickson's *History of the Theory of Numbers*, I believe that the last result in particular is new.

THEOREM I. *We have*

$$N(n) > \frac{a \cdot n}{\log n},$$

where a is a constant.

PROOF. For each prime p, $\phi(p) = p - 1$; hence, if we denote by $\pi(n)$ the number of primes not exceeding n, then

$$N(n) \geqq \pi(n).$$

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