AN ELEMENTARY PROOF BY MATHEMATICAL INDUCTION OF THE EQUIVALENCE OF THE CESÀRO AND HÖLDER SUM FORMULAS*

BY TOMLINSON FORT

For brevity, notation and terminology in this note are generally not explained. It is believed that they will be clear in all instances to any probable reader.

THEOREM. *The Hölder and Cesàro methods of summation are equivalent.*

PROOF. Let $C_n^{(2)}$ represent the Cesàro sum of order r to n terms:

$$(1)\qquad C_n^{(r)} = s_0\frac{r}{r+n} + \cdots + s_k\frac{r(n-k+1)\cdots n}{(r+n-k)\cdots(r+n)} + \cdots + s_n\frac{r(n!)}{r\cdots(r+n)}.$$

We readily verify that $C_n^{(r)}$ satisfies the equation

$$(2)\qquad (n+r+1)\,C_n^{(r+1)} - nC_{(n-1)}^{(r+1)} = (r+1)C_n^{(r)},$$

which may be written in the form

$$\Delta\left\{nC_{(n-1)}^{(r+1)}\right\} + r\,C_n^{(r+1)} = (r+1)C_n^{(r)},$$

or

$$(3)\qquad (n+1)C_n^{(r+1)} + r\sum_{n=0}^{n} C_n^{(r+1)} = (r+1)\sum_{n=0}^{n} C_n^{(r)}.$$

By solving (2) we get the following, as is easily verified:

$$C_n^{(r+1)} = \frac{n!}{(r+2)\cdots(r+n+1)}\sum_{n=0}^{n}\frac{(r+1)\cdots(r+n)}{n!}C_n^{(r)}$$

$$(4)\qquad = \frac{(r+1)!}{(n+1)\cdots(n+r+1)}\sum_{n=0}^{n}\frac{(n+1)\cdots(n+r)}{r!}C_n^{(r)}.$$

---

*Presented to the Society, December 29, 1926.

Let $H_n^{(r)}$ represent the Hölder sum to n terms of order r. We establish by mathematical induction the following fundamental formula.

$$
\begin{aligned}
H_n^{(r)} &= k_0^{(r)} C_n^{(r)} + k_1^{(r)} \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)} \\
&\quad + k_2^{(r)} \frac{1}{n+1} \sum_{n=0}^{n} \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)} + \cdots \\
&\quad + k_{r-1}^{(r)} \frac{1}{n+1} \sum_{n=0}^{n} \cdots \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)},
\end{aligned} \tag{5}
$$

where

$$k_0^{(r)} + k_1^{(r)} + \cdots + k_{r-1}^{(r)} = 1.$$

If we assume (5), we find

$$
\begin{aligned}
H_n^{(r+1)} &= \frac{1}{n+1} \sum_{n=0}^{n} H_n^{(r)} = k_0^{(r)} \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)} \\
&\quad + k_1^{(r)} \frac{1}{n+1} \sum_{n=0}^{n} \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)} \\
&\quad + k_2^{(r)} \frac{1}{n+1} \sum_{n=0}^{n} \frac{1}{n+1} \sum_{n=0}^{n} \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)} \\
&\quad + \cdots + k_{r-1}^{(r)} \frac{1}{r+1} \sum_{n=0}^{n} \frac{1}{n+1} \cdots \sum_{n=0}^{n} C_n^{(r)}.
\end{aligned} \tag{6}
$$

Substitute for

$$\frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)}$$

in each sum of (6), its value from (3). Take for example

$$
\begin{aligned}
k_1^{(r)} \frac{1}{n+1} \sum_{n=0}^{n} \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)} &= k_1^{(r)} \frac{1}{r+1} \cdot \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r+1)} \\
&\quad + k_1^{(r)} \frac{r}{n+1} \frac{1}{n+1} \sum_{n=0}^{n} \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r+1)}.
\end{aligned} \tag{7}
$$

We notice that

$$k_1^{(r)} \cdot \frac{1}{r+1} + k_1^{(r)} \frac{r}{r+1} = k_1^{(r)}.$$

We then notice that we have an expression of the same form as (5), but with r replaced by $(r+1)$. As $H_n^{(1)} = C_n^{(1)}$, proof of the formula follows by induction.

We next prove in a similar manner the formula

$$(8)\quad C_n^{(r)} = h_0^{(r)} H_n^{(r)}$$
$$+ h_1^{(r)} \frac{r!}{(n+1)\cdots(n+r)} \sum_{n=0}^{n} \frac{(n+1)\cdots(n+r-1)}{(r-1)!} H_n^{(r)}$$
$$+ h_2^{(r)} \frac{r!}{(n+1)\cdots(n+r)} \sum_{n=0}^{n}\sum_{n=0}^{n} \frac{(n+1)\cdots(n+r-2)}{(r-2)!} H_n^{(r)}$$
$$+ \cdots + h_{r-1}^{(r)} \frac{r!}{(n+1)\cdots(n+r)} \sum_{n=0}^{n} \cdots \sum_{n=0}^{n} (n+1) H_n^{(r)},$$

where

$$h_0^{(r)} + h_1^{(r)} + \cdots + h_{r-1} = 1.$$

To prove this formula, substitute in (4), and then make the substitution

$$(9)\quad \sum_{n=0}^{n} \frac{(n+1)\cdots(n+r-2)}{(r-2)!} H_n^{(r)}$$
$$= \frac{(n+1)\cdots(n+r-1)}{(r-2)!} \frac{1}{n+1} \sum_{n=0}^{n} H_n^{(r)}$$
$$- \sum_{n=0}^{n} \frac{(n+1)\cdots(n+r-2)}{(r-3)!} \cdot \frac{1}{n+1} \sum_{n=0}^{n} H_n^{(r)}$$
$$= (r-1) \frac{(n+1)\cdots(n+r-1)}{(r-1)!} H_n^{(r+1)}$$
$$- (r-2) \sum_{n=0}^{n} \frac{(n+1)\cdots(n+r-2)}{(r-2)!} H_n^{(r+1)},$$

and similarly for other sums. To prove (9), etc., sum by parts once, using the formula

$$\sum_{n=0}^{n} u(n)\Delta v(n) = u(n)v(n)\Big]_0^{n+1} - \sum_{n=0}^{n} v(n+1)\Delta u(n).$$

Notice that the coefficients in the last member of (9), namely $(r-1)$ and $-(r-2)$, add to unity, and then by mathematical induction formula (8) is proved.

The conclusions of the theorem are readily drawn from equations (5) and (8). Consider first that $C_n^{(r)} \to s$. Then by a repetition of the arithmetic mean theorem each sum in (5) approaches s and hence

$$H_n^{(r)} \to (k_0^{(r)} + k_1^{(r)} + \cdots + k_{r-1}^{(r)})s = s.$$

Next suppose that $H_n^{(r)} \to s$. We do not have a theorem so well known as the arithmetic mean theorem to refer to but, if we notice that when $H_n^{(r)}$ is replaced by a constant s, each sum in (8), as

$$\frac{r!}{(n+1)\cdots(n+r)} \sum_{n=0}^{n}\sum_{n=0}^{n} \frac{(n+1)\cdots(n+r-2)}{(r-2)!}s,$$

equals s, a little simple epsilon work gives the desired result that $C_n^{(r)} \to s$.

Difference equations from which the coefficients $k_j^{(r)}$ and $h_l^{(r)}$ can be calculated might be written down but are omitted as they are not necessary for the proof of the theorem.

HUNTER COLLEGE