INTEGERS REPRESENTED BY POSITIVE TERNARY QUADRATIC FORMS*

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1. *Introduction.* Dirichlet† proved by the method of §2 the following two theorems:

THEOREM I. $A=x^2+y^2+z^2$ *represents exclusively all positive integers not of the form* $4^k(8n+7)$.

THEOREM II. $B=x^2+y^2+3z^2$ *represents every positive integer not divisible by* 3.

Without giving any details, he stated that like considerations applied to the representation of multiples of 3 by B. But the latter problem is much more difficult and no treatment has since been published; it is solved below by two methods.

Ramanujan‡ readily found all sets of positive integers a, b, c, d such that every positive integer can be expressed in the form $ax^2+by^2+cz^2+du^2$. He made use of the forms of numbers representable by

$$\begin{aligned} A, B, C &= x^2+y^2+2z^2, \\ D &= x^2+2y^2+2z^2, \\ E &= x^2+2y^2+3z^2, \\ F &= x^2+2y^2+4z^2, \\ G &= x^2+2y^2+5z^2. \end{aligned}$$

He gave no proofs for these forms and doubtless obtained his results empirically. We shall give a complete theory for these forms. These cases indicate clearly methods of procedure for any similar form.

For a new theorem on forms in n variables, see §9.

* Presented to the Society, December 31, 1926.

† Journal für Mathematik, vol. 40 (1850), pp. 228–32; French translation Journal de Mathématiques, (2), vol. 4 (1859), pp. 233–40; Werke, vol. II, pp. 89–96.

‡ Proceedings of the Cambridge Philosophical Society, vol. 19 (1916–19), pp. 11–15. He overlooked the fact that $x^2+2y^2+5z^2+5u^2$ fails to represent 15.

2. *The Form B.* Let B represent a multiple of 3. Since -1 is a quadratic non-residue of 3, x and y must be multiples of 3. Thus $B=3\beta$, $\beta=3X^2+3Y^2+z^2$. Since $\beta\equiv 0$ or $1 \pmod 3$, β represents no integer $3n+2$. If β is divisible by 3, z is divisible by 3 and B is the product of a like form by 9. We shall prove that β represents every positive integer $3n+1$. These results and Theorem II give

THEOREM III. *$x^2+y^2+3z^2$ represents exclusively all positive integers not of the form $9^k(9n+6)$.*

We shall change the notation from β to f and employ the fact that the only reduced positive ternary forms of Hessian 9 are*

$$f = x^2 + 3y^2 + 3z^2, \qquad g = x^2 + y^2 + 9z^2,$$
$$h = x^2 + 2y^2 + 5z^2 - 2yz, \qquad l = 2x^2 + 2y^2 + 3z^2 - 2xy.$$

No one of g, h, l represents an integer $8m+7$. For g this follows from Theorem I, since $g\equiv A \pmod 8$. Suppose $l\equiv 7 \pmod 8$. Then z is odd, $2s\equiv 4 \pmod 8$, where $s=x^2+y^2-xy$. Thus s is even and $(1+x)(1+y)\equiv 1 \pmod 2$, x and y are even, and $s\equiv 0 \pmod 4$, a contradiction. Finally, let $h\equiv 7 \pmod 8$. If y is even, $h\equiv x^2+z^2 \quad 3 \pmod 4$. Hence y is odd and

$$3\equiv h\equiv x^2+(z-1)^2+1 \pmod 4,$$

so that x and $z-1$ are odd. Write $z=2Z$. Then $h\equiv 3+4Z(Z-1)\equiv 3 \pmod 8$.

Consider the ternary form lacking the term xy:

$$\phi = ax^2 + by^2 + cz^2 + 2ryz + 2sxz. \tag{1}$$

Its Hessian H is $a\Delta-bs^2$, where $\Delta=bc-r^2$. Take $H=9$, $s=1$, $\Delta=24t$, $t=6k+1$. Then $b=3\beta$, $\beta=8at-3$. If a is not divisible by 3, $\beta=48ak+8a-3$ is a linear function of k with relatively prime coefficients and hence represents an infinitude of primes.

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* Eisenstein, Journal für Mathematatik, vol. 41 (1851), p. 169. By the Hessian H of ϕ we mean the determinant whose elements are the halves of the second partial derivatives of ϕ with respect to x, y, z. Eisenstein called $-H$ the determinant of ϕ. The facts borrowed in this paper from Eisenstein's table have been verified independently by the writer.

Take $a=3n+1$. Then $\beta\equiv -1 \pmod 6$,

$$(2)\qquad \left(\frac{-3}{\beta}\right)=-1,\quad \left(\frac{2}{\beta}\right)=-1,$$
$$\left(\frac{t}{\beta}\right)=\left(\frac{\beta}{t}\right)=\left(\frac{-3}{t}\right)=1,\quad \left(\frac{-\Delta}{\beta}\right)=1.$$

Hence $w^2\equiv -\Delta \pmod \beta$ is solvable. We can choose a multiple r of 3 such that $r\equiv w \pmod \beta$. Then $(\Delta+r^2)/b$ is an integer c. Since ϕ represents $b\equiv 7 \pmod 8$, it is equivalent to no one of g, h, l and hence is equivalent to f. Thus f represents every $a=3n+1$.

THEOREM IV. *$x^2+3y^2+3z^2$ represents exclusively all positive integers not of the form* $9^k(3n+2)$.

This theory for B made use of forms of the larger Hessian 9. We shall next show how to deduce a theory making use only of forms having the same Hessian 3 as B.

3. *A New Theory for B.* We proved that f is equivalent to a form (1) having $a=3n+1$, $b=3\beta$, $r=3\rho$, $s=1$, where $\beta=8at-3$ is a prime. In $9=H=a(bc-r^2)-3\beta$, replace β by its value. Thence $c=(8t+3\rho^2)/\beta\equiv 1 \pmod 3$, $c=1+3\gamma$. In (1) replace x by $X-z$. We get

$$\psi = aX^2 - 6nXz + 3(n+\gamma)z^2 + 3\beta y^2 + 6\rho yz.$$

Write $3z=Z$, $3y=Y$. Then

$$3\psi = 3aX^2 - 6nXZ + (n+\gamma)Z^2 + \beta Y^2 + 2\rho YZ$$

is equivalent to $3f=3x^2+Y^2+Z^2$ and hence to B. In 3ψ, replace a by α, β by b, ρ by r. We conclude that (1) is equivalent to B if

$$(3)\qquad a=3\alpha,\qquad \alpha=3n+1,\qquad b=8\alpha t-3,$$
$$t=6k+1,\qquad s=-3n=1-\alpha.$$

We shall now give a direct proof that there exists a form (1) of Hessian 3 which satisfies conditions (3) and is equivalent to B. In $H=3$, replace a and s by their values in (3). We get

$$b+3+3\alpha r^2-\alpha bP=0,\qquad P=3c+2-\alpha.$$

Replace the first term b by its value in (3), and cancel α. We get

$$(4) \qquad 8t + 3r^2 - bP = 0, \qquad -24t \equiv (3r)^2 \qquad (\mathrm{mod}\ b).$$

This congruence is solvable by (2) with β replaced by b. By (4), $8t(1-P)\equiv 0$, $P\equiv 1 (\mathrm{mod}\ 3)$. Hence the value of c determined by P is an integer. The only two reduced forms of Hessian 3 are B and $\chi = x^2+2\sigma$, where $\sigma = y^2+yz+z^2$. Suppose $\chi\equiv 5 \pmod 8$. Then x is odd and $\sigma\equiv 2 \pmod 4$. Thus

$$(1+y)(1+z) \equiv 1, \quad y\equiv z\equiv 0 \ (\mathrm{mod}\ 2), \quad \sigma\equiv 0\ (\mathrm{mod}\ 4).$$

This contradiction shows that b is not represented by χ. Since (1) represents b, it is not equivalent to χ and hence is equivalent to B. Thus B as well as ϕ represents* $a=3\alpha$. This completes the new proof of Theorem III by using only forms of Hessian 3. The numbers represented by χ are given by Theorem XI.

4. *The Form* $C=x^2+y^2+2z^2$. By Theorem I, A represents every positive $4k+2$. Then just two of x, y, z are odd, say x and y, while $z=2Z$. Then $x=X+Y$, $y=X-Y$ determine integers X and Y. Hence

$$X^2+Y^2+2Z^2=2k+1,$$

so that C represents all positive odd integers.† If

$$m\neq 4^k(8n+7), m=X^2+Y^2+z^2,$$

by Theorem I. Hence

$$2m=(X+Y)^2+(X-Y)^2+2z^2.$$

Conversely, if C is even, it is of the latter form.

THEOREM V. *C represents exclusively all positive integers not of the form* $4^k(16n+14)$.

5. *The Form* $D=x^2+2y^2+2z^2$. If m is odd and $\neq 8n+7$, $m=x^2+Y^2+Z^2$ by Theorem I. Then $x+Y+Z$ is odd. Permuting, we may take x odd, and write $Y+Z=2y$, $Y-Z=2z$.

* If we apply the method of §2 when $H=3$ and hence take $s=1$, $b=3\beta$, where β is a prime $\equiv -1 (\mathrm{mod}\ 8)$, we find that it fails for all choices of Δ.

†Lebesque, Journal de Mathématiques, (2), vol. 2 (1857), p. 149, gave a long proof by the method of §2.

Then $m=D$. Next, let $m=2r$ be any even integer not of the form $4^k(8n+7)$. Then $r\neq 4^t(16n+14)$. By Theorem V, r is represented by C. Then $m=2r$ is represented by D with x even.

THEOREM VI. *D represents exclusively* all positive integers not of the form* $4^k(8n+7)$.

6. *The Form* $F=x^2+2y^2+4z^2$. Every odd integer is represented by C with $x+y$ odd, whence one of x and y is even. Any integer $\neq 4^k(8n+7)$ is represented by D, and $2D=(2y)^2+2x^2+4z^2$.

THEOREM VII. *F represents exclusively† all positive integers not of the form* $4^k(16n+14)$.

The simple methods used in proving Theorems V–VII apply also to $x^2+2^r y+2^s z$ when r and s are both $\leqq 3$, and when $r=1$ or 3, $s=4$.

7. *The Form* $G=x^2+2y^2+5z^2$. The only reduced forms of Hessian 10 are G, $J=x^2+y^2+10z^2$, $K=2x^2+2y^2+3z^2+2xz$, and $L=2x^2+2y^2+4z^2+2yz+2xz+2xy$. Neither J nor K represents a number of the form $2(8n+3)$. For, if K is even, $z=2Z$, $K=2M$, $M=X^2+y^2+5Z^2$, where $X=x+Z$. Since M is congruent to a sum of three squares modulo 4, it is congruent to 3 if and only if each square is odd, and then $M\equiv 7 \pmod 8$. If J is even, $x=y+2t$, $J=2N$, $N=(y+t)^2+t^2+5z^2\not\equiv 3 \pmod 8$.

We now apply the method of §2 to prove that G represents every positive integer prime to 10. Take $\Delta=16k$, $k=10l\pm 3$. Then $b=2\beta$, where $\beta=8ka-5$ represents infinitely many primes. Now

$$\left(\frac{-\beta}{k}\right)=\left(\frac{5}{k}\right)=\left(\frac{k}{5}\right)=-1,$$

$$\left(\frac{-\Delta}{\beta}\right)=-\left(\frac{k}{\beta}\right)=-\left(\frac{-\beta}{k}\right)=1.$$

Since (1) represents b, which is of the form $2(8n+3)$, it is not equivalent to J or K. Since it represents the odd a, it is not

* $D=x^2+(y+z)^2+(y-z)^2\neq 4^k(8n+7)$ by Theorem I.

† If F is even, x is even and F is the double of a form D.

equivalent to L. Hence (1) is equivalent to G, which therefore represents a.

If G represents a multiple of 5, it is the product of 5 by $g=5X^2+10Y^2+z^2$, whence G represents no $5(5n\pm 2)$. Also, g is divisible by 5 only when z is. Thus G is divisible by 25 only when it is a product of 25 by a form like G.

To prove* that G represents every 5α if $\alpha=5n\pm 1$ is odd, employ (1) with $a=5\alpha$, $b=2\beta$, $\beta=8\alpha t-5$, $r=2\rho$, $s=1\mp\alpha$. The Hessian of (1) is 10 if

$$\beta + 5 + 10\alpha\rho^2 - \alpha\beta P = 0, \qquad P = 5c \pm 2 - \alpha.$$

Take t prime to 10 and replace the first β by its value. Thus $8t+10\rho^2-\beta P=0$, $P\equiv\pm 1 \pmod 5$. Hence P yields an integral value for c. Also,

$$\left(\frac{t}{\beta}\right)=\left(\frac{-\beta}{t}\right)=\left(\frac{5}{t}\right),$$

$$(5) \qquad \left(\frac{5}{\beta}\right)=\left(\frac{\beta}{5}\right)=\left(\frac{\pm 8t}{5}\right)=-\left(\frac{t}{5}\right),$$

$$\left(\frac{-80t}{\beta}\right)=-\left(\frac{5t}{\beta}\right)=1.$$

Next, if G is even, $x=z+2w$ and $G=2T$, where

$$T = y^2 + 2w^2 + 2wz + 3z^2, \qquad S = x^2 + y^2 + 5z^2$$

are the only reduced forms of Hessian 5. Every positive integer a prime to 5 is represented by T. Take $\Delta=8k$, $k=10m\pm 1$. Then $b=a\Delta-5$ represents an infinitude of primes, and

$$\left(\frac{-2}{b}\right)=1,\ \left(\frac{-\Delta}{b}\right)=\left(\frac{k}{b}\right)=\left(\frac{-b}{k}\right)=\left(\frac{5}{k}\right)=\left(\frac{k}{5}\right)=1.$$

Now $b\equiv 3 \pmod 8$ is not represented by S, as proved for M. Hence (1) is equivalent to T and not S. Thus T represents a.

* Or we may use the method of §2. Of the ten properly primitive reduced forms of Hessian 50, all except g fail to represent numbers $\equiv 14 \pmod{16}$. To prove that g represents $\alpha=5n\pm 1$ when odd, take $\Delta=80t$, whence $b=10\beta$, $\beta=8\alpha t-5$; apply (5). From this proof was reconstructed the shorter one in the text.

We saw that $2T=G$ represents no $5(5m\pm 2)$. Thus T represents no $5(5n\pm 1)$. To prove that T represents every 5α, where $\alpha=5n\pm 2$, employ (1) with $a=5\alpha$, $s=1\pm 2\alpha$. Its Hessian is 5 if

$$b+5+5\alpha r^2-\alpha bP=0, \qquad P=5c-4\alpha\pm 4.$$

Replace the first term b by $8t\alpha-5$, where t is prime to 10. Thus $8t+5r^2-bP=0$. Hence $8t(1\mp 2P)\equiv 0$, $P\equiv\pm 3 \pmod 5$, and the resulting value of c is an integer. Also

$$\left(\frac{5}{b}\right)=\left(\frac{b}{5}\right)=\left(\frac{\pm 16t}{5}\right)=\left(\frac{t}{5}\right),$$

$$\left(\frac{t}{b}\right)=\left(\frac{-b}{t}\right)=\left(\frac{5}{t}\right), \quad \left(\frac{-40t}{b}\right)=1.$$

We have now proved the two theorems:

THEOREM VIII. *T represents exclusively all $\neq 25^k(25n\pm 5)$.*

THEOREM IX. *G represents exclusively all $\neq 25^k(25n\pm 10)$.*

8. *The Form $E=x^2+2y^2+3z^2$.* We shall outline a proof of the following theorem.

THEOREM X. *E represents exclusively all $\neq 4^k(16n+10)$.*

The only reduced forms of Hessian 6 are E and

$$Q=x^2+y^2+6z^2, \qquad R=2x^2+2y^2+2z^2+2xy.$$

To prove that E represents every positive odd integer a, take $\Delta=9k$, $k=8t+3$. Then $b=3\beta$, where β represents primes. Also $(-\Delta/\beta)=1$. The resulting form (1) represents the odd a and hence is not equivalent to R. Since it represents $b=3(3n+1)$, it is not equivalent to Q. For, if Q is divisible by 3, both x and y are.

If E is even, then $x=z+2t$ and $E=2U$,

$$U=y^2+2z^2+2zt+2t^2$$

and conversely. In place of U we employ the like form χ of §3. To show that χ represents $a=2\alpha$ when α is odd, take $\Delta=9k$, $k=4t+1$. Then $b=a\Delta-3\equiv 6 \pmod 9$ is not represented by the remaining reduced form B of Hessian 3 (Theorem III). Also $b=3\beta$, $(-\Delta/\beta)=+1$.

Finally, χ represents every positive odd integer $a \not\equiv 5 \pmod 8$. Write $\alpha = \frac{1}{2}(3a-1)$ and take $\Delta = 9k$, $k = 2h+1$. Then $b = 6q$, $q = 3ah+\alpha$. If $a = 8A+1$, take $h = 4t$, t odd. Then $q = 12Ak + 12t+1$,

$$\left(\frac{-\Delta}{q}\right) = \left(\frac{k}{q}\right) = \left(\frac{q}{k}\right) = \left(\frac{12t+1-k}{k}\right) = \left(\frac{4t}{k}\right) = \left(\frac{k}{t}\right) = 1.$$

If $a = 8A+3$, take $h = 4t+1$. If $a = 8A+7$, take $h = 4t+1$. In each case $(-\Delta/q) = 1$. In all three cases, q represents an infinitude of primes.

THEOREM XI. *$x^2+2y^2+2yz+2z^2$ represents exclusively all positive integers not of the form $4^k(8n+5)$.*

9. *Forms in n Variables.* By a simple modification of Ramanujan's determination of quaternary forms which represent all positive integers, we readily prove*

THEOREM XII. *If, for $n \geqq 5$, $f = a_1x_1^2 + \cdots + a_nx_n^2$ represents all positive integers, while no sum of fewer than n terms of f represents all positive integers, then $n = 5$ and*

$$f = x^2 + 2y^2 + 5z^2 + 5u^2 + ev^2, \quad (e = 5, 11, 12, 13, 14, 15),$$

and these six forms f actually have the property stated.

After this paper was in type, I saw that J. G. A. Arndt gave† the Dirichlet type of proof which appears in §2 above, but not the improved new proof of §3. For the form G of §7, he treats only numbers not divisible by 5.

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*Note to appear in Proceedings of the National Academy.

† *Ueber die Darstellung ganzer Zahlen als Summen von sieben Kuben*, Dissertation, Göttingen, 1925.