ON A PROBLEM IN CLOSURE*

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The following note concerns finite groups of birational transformations which leave an algebraic curve of genus 1 invariant.

Let the curve be expressed as a C_4 in S_3, intersection of two general quadric surfaces. This curve is invariant under the linear group G_8, of order eight, generated by the harmonic homologies defined by the self-conjugate tetrahedron associated with C_4. The points of C_4 are thus arranged in sets of 8, forming a linear $I_8{}^1$ of genus 0. If the curve be projected upon a plane from an arbitrary point, a plane quartic C_4 results with nodes at K_1, K_2. The four central homologies become four perspective quadratic involutions T_i with centers O_i not on C_4; the three axial involutions become three non-perspective quadratic inversions, with fundamental points O_{ik} for $T_iT_k = T_kT_i = T_{ki} = T_{lm}$ at the diagonal points of the quadrangle $O_1O_2O_3O_4$. The nodes K_1, K_2 are the other fundamental points for all seven operations.†

From the theorem of Bertini it follows that in any plane nodal quartic one and only one conic can be found meeting

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† A brief synthetic outline of the properties of G_8 was first given by C. Segre, *Su una trasformazione irrazionale dello spazio . . .* , Giornale di Matematiche, vol. 21 (1883), pp. 355–378. This was amplified in connection with a larger problem by D. Montesano, *Su alcuni gruppi chiusi di trasformazioni involutorie nel piano e nello spazio*, Atti Istituto Veneto, (6), vol. 6 (1888), pp. 1425–1444. It is also contained in the papers by K. Meister, *Ueber die Systeme, welche durch Kegelschnitte mit einem gemeinsamen Polardreieck, bez. durch Flächen zweiten Grades mit einem gemeinsamen Polartetraeder gebildet werden*, Zeitschrift der Mathematik und Physik, vol. 31 (1886), pp. 321–347; vol. 34 (1889), pp. 6–24; 73–91 and by H. E. Timerding, *Ueber die quadratische Transformation durch welche die Ebenen des Raumes in ein System von Flächen zweiter Ordnung mit gemeinsamen Poltetraeder übergeführt werden*, Annali di Matematica, (3), vol. 1 (1898), pp. 95–117.

any line through the node in two points which form with the residual intersections of the line and the quartic a harmonic set. Using this conic as conic of invariant points and the node as center in a perspective quadratic inversion, the quartic is transformed into itself. In case there are two nodes there is a conic of Bertini associated with each, and the conic associated with one node passes through the other. Let these operations be denoted by S_1, S_2. In a recent memoir* the question is raised whether the group generated by S_1, S_2 and G_8 can be finite, but the author does not answer it. The answer may be difficult if the method is restricted to pure geometry. By assistance of the parametric representation in terms of elliptic functions it can be answered in every case.

Let

$$F = x_1^2 - x_2^2 + x_3^2 - x_4^2 = 0, \qquad \sum a_i x_i^2 = 0$$

define the curve. The tetrahedron of reference is the self-conjugate tetrahedron of the pencil of quadrics, and the equations of the harmonic homology H_1 with center at vertex (1, 0, 0, 0), invariant plane $x_1=0$, are $px_1' = -x_1$, $px_i' = x_i$, $i=2, 3, 4$; similarly for H_2, H_3, H_4.

By projecting $F=0$ upon $x_4=0$ from (1, 1, 1, 1), the operation H_1 becomes the perspective inversion

$$(T_1) \qquad \begin{cases} x_1' = (x_2 - x_3)(x_1 - x_2 - x_3), \\ x_2' = x_2(x_1 - x_2 + x_3), \\ x_3' = x_3(x_1 + x_2 - x_3). \end{cases}$$

The center is $O_1 = (1, 0, 0)$, and the other fundamental points are

$$K_1 = (0,1,1), \qquad K_2 = (1,1,0).$$

The conic of invariant points is

$$(x_1 - x_2)^2 + 2x_1x_3 - x_3^2 = 0.$$

* E. Ciani, *Le quartiche piane invertibili*, Giornale di Matematiche, vol. 57 (1919), 47 pages.

Similar expressions exist for T_i, $i=2, 3, 4$, and for T_{ik}, $O_4=(1, 1, 1)$. The triangle formed by any three of the points $O_i\ O_k\ O_l$ is self-conjugate as to the conic of invariant points associated with the fourth.*

Among the space quartics belonging to the same self-conjugate tetrahedron are ∞^1 through the center of projection. This pencil is projected into a pencil of plane cubics through K_1, K_2, the four vertices O_1, O_2, O_3, O_4, and the three diagonal points O_{ik}. Hence by suppressing any side of the quadrangle, the four remaining basis points and K_1, K_2 lie on a conic.

The equation can be reduced to the form

$$k_1x_1(x_2-x_3)(x_2+x_3-x_1)+k_2x_2(x_3-x_1)(x_1-x_2+x_3)=0.$$

The points K_1, K_2 form a pair of conjugate points in the Geiser net determined by the other seven basis points. The line joining them meets any C_3 of the pencil in just one point, which with the seven basis points makes a group of I_8 on that curve. Any position on C_3 can be chosen for one of them, and then the other is uniquely fixed.

The points O_i are all cotangential, and any one of them can be assumed at will. The fundamental points O_{ik} are collinear with O_iO_k and with O_lO_m. Associated with any point P on C_3 are the points of tangency of the four tangents from P. The other three from the first tangential of P have their points of contact at O_{ik}. Expressed in parametric form, assuming the form to which every elliptic cubic can be reduced, that three points are collinear when the sum of their parameters is congruent to zero, then if P has the parameter u, we have

$$O_1 = -\frac{u}{2}, \qquad O_2 = -\frac{u}{2}+\omega, \qquad O_3 = -\frac{u}{2}+\omega',$$

$$O_4 = -\frac{u}{2}+\omega+\omega', \qquad O_{12} = u+\omega, \qquad O_{13} = u+\omega',$$

$$O_{14} = u+\omega+\omega'.$$

* The equations of the operations of the group and those of an ∞^2 family of binodal quartics each invariant under it were determined in my seminar by Miss Bertha I. Hart, of Western Maryland College.

Let K_1 have the parameter v; then K_2 has the parameter $-u-v$. A conic through K_1 and through the four points remaining of the seven basis points when three on any line have been suppressed will also pass through K_2.

The involution in which K_1, K_2 are a pair of conjugate points is that having the polar conic at P for conic of invariant points, and P for vertex.

If K_1, K_2 be regarded as fixed, then a cubic of the pencil is fixed by its residual intersection with K_1, K_2. Since $P_1 K_1$ can be chosen at will and satisfy all the conditions of the problem it follows that the group generated by K_1, O_1 is generally of infinite order. If finite, K_1, G_8 is also finite.

Since P can be chosen at will on C_3 and K_1 can be chosen at will, there are ∞^2 groups of I_8 in the plane. As groups of operations there appear in sets of pencils, those of any set (O_i on any curve of the pencil) having the three operations T_{ik} in common, so far as the transformations of points on the curve are concerned.

These results can be interpreted in terms of C_4 in space directly. By an appropriate linear transformation we may write*

$$x_1=\sigma_1(2u), \quad x_2=\sigma_2(2u), \quad x_3 = \sigma_3(2u), \quad x_4=\sigma(2u),$$

the σ_i being those defined in the Schwarz-Weierstrass *Formeln und Lehrsätze*, pp. 21–2.

Four points with parameters $u_1, \cdots, u_4$ are collinear if $\Sigma u_i=0$. The operations of G_8 are defined as follows:

$$\begin{aligned} u' &= -u + \alpha + \omega' && \text{is } H_1, \\ u' &= -u + \omega' && \text{is } H_2, \\ u' &= -u + \omega && \text{is } H_3, \\ u' &= -u && \text{is } H_4, \end{aligned}$$

from which the H_{ik} at once follow.

A quadric of the pencil through C passes through any given point P. The quadric contains a generator of each system,

* E. Lange, *Die sechzehn Wendeberührungspunkte der Raumcurve vierter Ordnung, erster Species*, Schlömilchs Zeitschrift, vol. 28 (1883), pp. 1–23.

through P, each meeting C_4 twice. If this quadric be projected stereographically from P, the plane quadric will have a double point at K_1 on one generator and at K_2 on the other. Consider the line PK_1. It contains P_1, P_2 on C_4 and any plane through PK_1 meets C_4 in two points P_3, P_4 which project into two points collinear with K_1. These points are interchanged by S_1, hence in space we may choose u_1 and u_2 at will, then pass a plane through the points determined by them and any third point u_3. The space operation consists in interchanging the point u_3 with the fourth point u_4 as the plane turns about u_1 u_2, as u_3 describes C_4.

The operation has the form

$$u' = -u + c,$$

where c is any given constant. If this be associated with H_i the product is not periodic unless c is of the form $r\omega + r'\omega'$ wherein r, r' are both rational. But if this product is rational, then the group generated by S and G_8 is finite. The operation S_2 is defined by

$$u' = -u - c,$$

hence the necessary and sufficient condition that S_1, S_2, G_8 generate a finite group is that S_1, S_2 generate a finite group.

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