ON THE METRIZATION PROBLEM AND RELATED PROBLEMS IN THE THEORY OF ABSTRACT SETS*

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1. *Topological Space.* In the theory of abstract sets we assume that we are given an arbitrary aggregate P and a relation between subsets of P which corresponds to the relation between a set and its derived set in the classical theory of sets of points.† That is, the mathematical concept abstract set in its current sense includes the notion limit point or point of accumulation. The introduction of limit points permits the definition of continuous 1-1 correspondence or homeomorphy. The study of such correspondences, particularly of invariants under homeomorphic transformations, constitutes the science of topology or analysis situs.‡ It seems proper therefore to speak of an abstract set as a topological space.§ Throughout this paper, the term *topological space* or *abstract set* refers to any system of the form (P, K) composed of an aggregate P and a relation of the form EKE' between the subsets E, E' of P which is subject to the condition, for every subset E of the aggregate P there is a unique set E' in the relation K to E. That is, the relation K defines a single-valued set-valued function on the class U of all subsets of the aggregate P.‖

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† See M. Fréchet, *Esquisse d'une théorie des ensembles abstraits*, Sir Asutosh Mookerjee's Commemoration volumes, II, p. 360, The Baptist Mission Press, Calcutta, 1922; *Sur les ensembles abstraits*, Annales de l'Ecole Normale, vol. 38 (1921), p. 341ff.

‡ See H. Tietze, *Beiträge zur allgemeinen Topologie*, I., Mathematische Annalen, vol. 88 (1923), p. 290.

§ This terminology is suggested by Fréchet. See Comptes Rendus, vol. 180 (1925), p. 419.

‖ These functions are studied in detail in an unpublished article by the writer.

2. *The Metrization Problem.* The problem is to state in terms of the concepts point, and point of accumulation the conditions that a topological space be metric.

A metric space* is any topological space in which the points of accumulation are defined or definable in terms of a function (p, q) called the distance between the points p and q and satisfying the following conditions.

(0) *The distance (p, q) is a definite real number for every pair of points p, q.*

(1) *Two points are coincident if and only if their distance is zero.*

(2) *For any three points p, q, r,*

$$(p,q) \leqq (p,r) + (q,r).$$

It follows readily from these conditions that the distance (p, q) is non-negative, and that it is symmetric in p and q, $(p, q) = (q, p)$.† In a metric space a point p is a point of accumulation of a set E provided its distance from a variable point of E which is distinct from p has the lower bound zero.

The following illustrations convey some notion of the scope of the concept metric space. If the aggregate P denotes the linear continuum of all real numbers and $(p, q) = |p-q|$, the resulting space is metric. Similarly euclidean space is also metric. The Hilbert‡ space of infinitely many dimensions in which the coordinates $x_1, x_2, x_3, \cdots, x_n, \cdots$ of each point are subject to the condition that the sum of their squares be a convergent series is a metric space in which distance is defined by the formula

$$(p,q) = [(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2 + \cdots]^{1/2},$$

* F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, 1914, p. 211. The definition of distance is due to Fréchet, *Sur quelques points du calcul fonctionnel*, Rendiconti di Palermo, vol. 22 (1906).

† A. Lindenbaum, *L'espace métrique*, Fundamenta Mathematicae, vol. 8 (1926), pp. 209–222.

‡D. Hilbert, Göttinger Nachrichten, vol. 8 (1906). Hausdorff, loc. cit., p. 287.

in which the x_n, y_n are the coordinates of p and q, respectively.

If we take for our aggregate P the class of all functions $p=f(x)$, on an interval $a \leqq x \leqq b$, whose squares are summable, and define distance by the formula

$$(p,q) = \left[\int_a^b [f(x) - g(x)]^2 \, dx\right]^{1/2},$$

we obtain a metric space. It is necessary to make the convention that two points coincide if the corresponding functions differ only at a set of points of measure zero. The relation $p=Lp_n$ corresponds to convergence in the mean for the sequence of functions $p_n=f_n(x)$.

3. *Hausdorff Spaces.* A remarkable and important class of topological spaces has been defined by F. Hausdorff.* In a Hausdorff space the points of accumulation are defined in terms of a family of neighborhoods U conditioned by the following four postulates.

(A) *To every point p there corresponds at least one neighborhood U, and each neighborhood of p contains p.*

(B) *There is a neighborhood of a point p common to every two neighborhoods of p.*

(C) *If q is any element of a neighborhood U of a point p, then U contains all the points of a neighborhood of q.*

(D) *If p and q are distinct points, there exist neighborhoods of p and q, respectively, which have no common elements.*

The topological spaces of Hausdorff are evidently included among the classes (V) of Fréchet† in which the postulated family of neighborhoods is subject only to the condition (A). In a class (V) a point p is a point of accumulation of a set E in case every neighborhood of p contains a point of E distinct from p.

* Loc. cit., p. 212.

† *Esquisse*, p. 346; Bulletin des Sciences Mathématiques, (2), vol. 42 (1918), pp. 1–19.

It is easy to show that every metric space is a Hausdorff space. Let $S(p, a)$ denote the generalized sphere composed of all the points q of a metric space which satisfy the inequality $(p, q) < a$. The family of all such spheres of center p and radius a has the four properties of Hausdorff and evidently defines the same points of accumulation as the distance (p, q).

The following set of properties form a necessary and sufficient condition that a topological space be a Hausdorff space.*

(I) $(A+B)' = A' + B'$.

(II) *A set containing but a finite number of points has no point of accumulation.*

(III) *The derived set of every set is closed.*

(IV) *If p and q are any two distinct points there exist open sets U and V which are disjoined and contain p and q respectively.*

To complete the solution of the metrization problem we need only add the conditions that a Hausdorff space be metric.

4. *Existence of Non-Constant Continuous Functions.* The metrization problem is included in another problem proposed by Fréchet in correspondence with Paul Urysohn and with me. It is evident that the distance (p, q) of two points p and q is a continuous function of its arguments, and is not constant in a space of two or more points. Thus the metrization problem is related to the more general problem, under what conditions does a topological space admit the existence of a non-constant continuous function.† The topological conditions for the existence of such functions in a Hausdorff space have been dis-

* *Esquisse*, p. 367.

† The definition of continuous function for general topological space is given by Fréchet in the following form: *A point p of space is interior to a set I if it belongs to I and is not a point of accumulation of any subset of the of the complement of I, $P-I$. A function $f=f(p)$ is continuous at a point p if the oscillation of the function f on the sets I to which p is interior has the lower bound zero. Esquisse*, p. 363.

covered by Urysohn.* I have recently succeeded in formulating these conditions for topological spaces in general.

The following form of the question regarding the existence of non-constant continuous functions is of particular importance in the present discussion. Characterize those spaces of Hausdorff in which it is possible to define for every two disjoined closed sets A and B a continuous function $f(p)$ which is equal to zero on A, one on B, and satisfies the inequality

$$0 \leqq f(p) \leqq 1$$

everywhere else, that is, on the set $C=P-A-B$.

We shall show that it is both necessary and sufficient for the space to be *normal.* A space is normal provided every pair of closed disjoined sets A and B is separated by open sets, that is, that there exist open sets U, V which are disjoined and include A and B respectively.

Since the condition is evidently necessary we proceed to the proof of its sufficiency. Let P be a normal Hausdorff space and let A and B be any two disjoined subsets of P. From the hypothesis of normality there exist two open sets U_1, V_1 such that

$$A = U_1, \qquad B = V_1, \qquad U_1V_1 = 0.$$

The set $P-U_1$ is closed and includes V_1. It follows that there exist disjoined open sets U_0, V_0 such that

$$A \leqq U_0, \qquad P - U_1 \leqq V_0.$$

Therefore†

$$A \leqq U_0 \leqq U_0^0 \leqq U_1, \qquad B \leqq P - U_1.$$

Since U_0^0 and $P-U_1$ are disjoined closed sets there is an open set $U_{1/2}$ for which

$$A \leqq U_0^0 \leqq U_{1/2} \leqq \overset{0}{U}_{1/2} \leqq U_1.$$

* *Über die Mächtigkeit der zusammenhängenden Mengen,* Mathematische Annalen, vol. 94 (1925), p. 290.

† $U^0 = U + U'$.

It follows readily that there exists for any positive integer n a series of the form

$$A \leqq U_0^0 \leqq U_{1/2^n} \leqq \overset{0}{U}_{1/2^n} \leqq \cdots \leqq U_{m/2^n} \leqq \overset{0}{U}_{m/2^n} \leqq \cdots \leqq U_1$$

where $B=P-U_1$.

If $x=Lr_n$, where $r_n=m_n/2^n$, and $0\leqq r_n \leqq r_{n+1} \leqq x \leqq 1$, we set

$$U_x = \sum_{n=1}^{\infty} U_{r_n}.$$

By definition $U_x \leqq U_{x'}$ if $x<x'$. Furthermore $U_x^0 \leqq U_{x'}$. For if n and m are chosen so that $x<m/2^n<(m+1)/2^n<x'$, then

$$U_x^0 \leqq \overset{0}{U}_{m/2^n} \leqq \overset{0}{U}_{(m+1)/2^n} \leqq U_{x'}.$$

Let $L_0=U_0$; $L_1=P-U_1$; $L_x=U_x^0-U_x$, $0<x<1$. Then the sets $L_x(0\leqq x\leqq 1)$ are closed, and if $x \neq x'$, $L_xL_{x'}=0$. The function $f(p)$ which is equal to x when p is a point of L_x has the required properties.

5. *Perfectly Separable Spaces.* Among the spaces which were considered by Hausdorff are those whose points of accumulation are definable in terms of an enumerable family of neighborhoods.* Such spaces are said to satisfy the second axiom of enumerability. Tychonoff and Vedenissof have called them separable spaces.† In a letter to me Fréchet calls attention to the fact that the word separable is already in use in a more general sense and suggests the term perfectly separable. The following important and remarkable theorem was discovered by Urysohn.‡

THEOREM. *A necessary and sufficient condition that a perfectly separable Hausdorff space be metric is that it be normal.*

It is easy to show that every metric space is normal.§ Let A and B be any two disjoined closed sets and let a_p denote the

* Loc. cit., p. 263.

† Bulletin des Sciences Mathématiques, vol. 50 (1926), p. 17.

‡ Mathematische Annalen, vol. 94 (1925), p. 309.

§ Tietze, loc. cit., p. 311.

lower bound of the distance (p, q) as q varies over B. Let U be the sum of the spheres $S(p, a_p/3)$ obtained by considering all the points of A. Let V denote the corresponding open set enclosing B. Then U and V are disjoined. For if r is a point common to U and V there exist points p and q such that $(p, r) < c/3$, $(q, r) < c/3$, where c is the greater of a_p, b_q. Then we have

$$(p, q) \leqq (p, r) + (r, q) < 2c/3,$$

contrary to the befininition of a_p b_q.

To show that the condition is sufficient let

$$U_1, \quad U_2, \quad U_3, \cdots, U_i, \cdots, U_j, \cdots$$

represent an enumerable set of neighborhoods determining the points of accumulation of a *normal* Hausdorff space P. For each of the enumerable family of pairs of neighborhoods U_i, U_j, such that $U_i^0 \leqq U_j$, there is a continuous function $f = f(p)$ which satisfies the conditions $f(p) = 0$ on U_i^0, $f(p) = 1$ on $P - U_j$, $0 \leqq f(p) \leqq 1$ on P. Let the set of all such functions be represented by the sequence

$$f_1, f_2, f_3, \cdots, f_n, \cdots.$$

Consider the function

$$(p, q) = \sum_{n=1}^{\infty} [f(p) - f(q)]/2^n.$$

This function is continuous because it is the sum of a uniformly convergent series of continuous functions. It is furthermore evident that $(p, p) = 0$, and that $(p, q) \leqq (p, r) + (q, r)$.

It is necessary to show that if $p \neq q$ then $(p, q) > 0$. Let U_j be a neighborhood of p which does not contain the point q. Since the sets $A = p$ and $B = P - U_j$ are closed and disjoined there is a neighborhood U_i such that $U_i^0 = U_j$. Let $f_n(p)$ be the continuous function corresponding to the two sets U_i, U_j. Since $f_n(p) = 0$ and $f_n(q) = 1$, we have $(p, q) \geqq 1/2^n$, which was to be proved.

It remains to show that the neighborhoods U_i and the distance (p, q) just defined determine the same space. This will be the case if for each point p every neighborhood of p contains a sphere $S(p, a)$ and if every such sphere contains a neighborhood of p. Since the distance (p, q) is continuous in q it follows that a neighborhood of p, U_i, can be found on which the oscillation of the function (p, q) is less than $a/2$ for any value of a (>0). Hence the sphere $S(p, a)$ includes the neighborhood U_i.

Suppose that there is a point p and a neighborhood U_j of p such that for every positive value of a there is a point q_a of the sphere $S(p, a)$ which is exterior to U_j. As before there exists a neighborhood U_i of p such that $U_i \leqq U_j$. Let f_n be the function associated with the pair of neighborhoods U_i, U_j. Then $f_n(q) = 1$, and $(p, q) = 1/2^n$. This is impossible if $a < 1/2^n$.

6. *Axioms of Separation.* The property of normality is the third of a series of four axioms of separation which have been discussed in detail by Tietze.* Two point sets A and B are separated by open sets U, V if

$$A = U, \qquad B = V, \qquad UV = 0.$$

We consider the following four cases.

(1) *The sets A and B each contain one point only.*

(2) *The set A contains a single point, the set B is closed.*

(3) *A and B are any closed sets.*

(4) *A and B are disconnected. That is, neither set contains a point of accumulation of the other.*

The first of these axioms of separation is Axiom (D) of the set defining a Hausdorff space, and coincides with the fourth of the conditions that a topological space be a space of Hausdorff. We indicate the axiom which corresponds to case i by the symbol (IV_i). In the terminology of Paul Alexandroff and Urysohn† a space satisfying axiom (IV_2) is regular; (IV_3) is normal; (IV_4) is completely normal. Each of these axioms

* Loc cit., p. 300, etc.

† Mathematische Annalen, vol. 92 (1924), p. 263.

is stronger than its predecessor and independent of it. Metric spaces are completely normal. However there exist spaces which are completely normal but not metric.*

7. *Regular Spaces.* Is a regular perfectly separable Hausdorff space normal? This question was proposed by Urysohn† and answered in the affirmative by Tychonoff.‡ Let A and B be any disjoined closed sets and let

$$U_1, U_2, U_3, \cdots, U_m, \cdots$$

be an enumerable system of open sets defining the points of accumulation of a Hausdorff space P. For each point p of the set A there is a neighborhood U_{n_p} of p which with its derived set contains no point of B. The class of all the neighborhoods U_{n_p} determined by the points of A and the set B forms a sequence

$$V_1, V_2, V_3, \cdots, V_n, \cdots$$

of open sets whose sum includes the set A. In similar fashion we define a sequence of open sets

$$W_1, W_2, W_3, \cdots, W_n, \cdots$$

hose sum includes B, such that $W_n^0 A = 0$, $n = 1, 2, 3, \cdots$. Let $G_1 = U_1$, $H_1 = V_1 - G_1^0$, and in general,

$$G_n = U_n - \sum_{i=1}^{n-1} H_i^0, \qquad H_n = V_n - \sum_{i=1}^{n} G_i^0.$$

The sets G_n and H_n are evidently open. If we now set

$$G = \sum_{n=1}^{\infty} G_n, \qquad H = \sum_{n=1}^{\infty} H_n,$$

we can show that

$$A \leqq G, \qquad B \leqq H, \qquad GH = 0.$$

Since no point of A is contained in any set V_n, $G_n A = U_n A$, and therefore $A \leqq G$. Similarly $B \leqq H$. Suppose that there is a point p common to G and H. Then there must exist indices

* Tietze, loc. cit.

† Mathematische Annalen, vol. 94 (1925), p. 315.

‡ Mathematische Annalen, vol. 95 (1926), p. 139.

m and n for which p is common to the sets G_m, H_n. If $m \leqq n$, we have

$$G_m H_n = G_m\left(V_n - \sum_{i=1} G_i^0 \right) \leqq G_m(V_n - G_m^0) = 0,$$

a contradiction. A similar contradiction is obtained if $m > n$.

This result when combined with the theorem of the preceding section gives the fundamental theorem.

THEOREM. *A necessary and sufficient condition that a perfectly separable Hausdorff space be metric is that it be regular.*

8. *An Axiom of R. L. Moore.* The importance of the regular and perfectly separable, therefore metric, spaces in the analysis of continua is indicated by the fact that nine years before the publication of the discoveries of Urysohn, R. L. Moore assumed these properties in the first of a system of axioms for the foundations of plane analysis situs.* This axiom is furthermore of particular interest historically since it yields when slightly modified a necessary and sufficient condition that a topological space be metric and separable. The modified axiom of Moore may be stated as follows.

AXIOM (R. L. Moore). *We are given a space P in which point of accumulation is defined in terms of a family of classes of points called regions. Among the regions there exists a fundamental enumerable sequence*

$$R_1, R_2, R_3, \cdots, R_n, \cdots$$

with the following properties: (0) *for every region R there is an integer n such that R_n is a subset of R*; (1) *for every point p and integer n there is an integer n' greater than n such that $R_{n'}$ contains p*; (2) *if p and q are distinct points of a region R there is an integer m such that if n is greater than m and R_n contains p, then R_n is a subset of $R - q$.*†

* *On the foundations of plane analysis situs*, Transactions of this Society, vol. 17 (1916), pp. 131–164. The hypothesis of regularity was also made (apparently independently) by L. Vietoris, Monatshefte, vol. 31 (1921), p. 176.

† This differs from Axiom 1 (loc. cit.) with respect to the first sentence and condition (0) only.

It is easy to show that in a space satisfying this axiom the regions are open sets, and that the space is a regular and perfectly separable Hausdorff space, therefore metric and separable. The converse of this proposition is also true and is established by the following chain of propositions.

The axiom of Moore is a topological invariant. That is, if the axiom is satisfied in one of two homeomorphic spaces it is satisfied in the other. Furthermore if the axiom holds in a space P it holds in any relative subspace of P.

It is easy to show that every compact metric space admits the axiom. Since Urysohn* has shown that every separable metric space is homeomorphic with a subset of a compact domain in the Hilbert space it follows immediately that every separable metric space satisfies this axiom.†

THEOREM. *A necessary and sufficient condition that a topological space be metric and separable is that it satisfy the axiom of R. L. Moore.*‡

9. *Metrization of Compact Spaces.* It can now be shown that a compact and perfectly separable Hausdorff space is metrizable. It is sufficient to show that it is regular. We have to show that for each point p and closed set A there are open sets V, U which separate p and A. Enclose each point of a given closed set A in an open set which does not contain p and consider the system of open sets thus obtained together with the open set $P-A$. Since every compact perfectly separable space has the "any-to-finite"§ property of Borel, a finite subset of these open sets may be selected which covers the closed set A. Let

$$U_1, U_2, U_3, U_4, \cdots, U_n$$

* Mathematische Annalen, vol. 92 (1924), p. 302.

† I have obtained a direct proof of this proposition.

‡ The fact that Axiom 1 is a sufficient condition for metrizability was inferred by R. L. Moore from the theorem of Tychonoff in §7 above.

§ The phrase was introduced by T. H. Hildebrandt in an article on *The Borel theorem and its generalizations*, this Bulletin, vol. 32 (1926), pp. 423–474.

be a finite family of open sets whose sum U includes A but not p. Then there is a neighborhood V of p which contains no point of the open set U. The proof of the following theorem of Urysohn* is now easily completed.

THEOREM. *A necessary and sufficient condition that an infinite compact Hausdorff space be metrizable is that it be perfectly separable.*

10. *The Metrization Conditions in Locally Compact Spaces.* The result just obtained has been applied by Alexandroff† to locally compact Hausdorff spaces. A space is locally compact provided there is for every point p a neighborhood V such that V^0 is compact. The following fundamental theorem may be stated.

THEOREM. *A necessary and sufficient condition that a locally compact Hausdorff space be metrizable is that the space be perfectly separable or else be the sum of a set (of arbitrary cardinal number) of disjoined domains which are perfectly separable subspaces of the given space.*

The proof that this condition is sufficient follows lines indicated previously. The essential part of the proof that the condition is necessary consists in showing that every locally compact metric space which is not perfectly separable is representable as a sum of disjoined perfectly separable spaces.

In a locally compact metric space every point p is the center of a sphere $S(p, a)$ which is compact and therefore perfectly separable. Let $G_0 = S(p_0, a_0)$ and assume that G_n is defined and perfectly separable. Then G_{n+1} is defined to be the sum of all the spheres $S(p, a)$ which contain a point of G_n. The set

$$G = G_0 + G_1 + \cdots + G_n + \cdots$$

is perfectly separable. If p_0 is replaced by any point of G the process just defined will lead to the same set G. Thus the

* Mathematische Annalen, vol. 92 (1924), pp. 275–293, and vol. 94 (1925), p. 313.

† Mathematische Annalen, vol. 92 (1924), pp. 294–301.

given space admits a decomposition into disjoined perfectly separable metric subspaces.

This theorem has the following interesting corollary.

COROLLARY. *A necessary and sufficient condition that a connected and locally compact space be metrizable is that it be perfectly separable.*

11. *Relations between Metrization and the Existence of Continuous Functions.* We have seen that the metrization problem is related to the more general problem of the existence of non-constant continuous functions and how the study of the latter problem led Urysohn to the solution of the metrization problem for perfectly separable spaces. It is therefore of interest to formulate the conditions that a space be metrizable in terms of continuous functions.

THEOREM. *A necessary and sufficient condition that a topological space which is equivalent to a class (V) of Fréchet be metrizable is that there exist a family of equally continuous functions with the following properties:*

(1) *each function of the family is defined and continuous throughout the space;*

(2) *for each point p there is a function of the family which vanishes at p and is bounded from zero on the complement of every neighborhood of p.*

To show that the condition is necessary, let P be a metric space and define the required family of functions to be the class of all functions $\phi(p)=(p, q)$, where (p, q) is the distance from p to q, the point q is held fixed and the point p allowed to vary. That this family of functions has the required properties is an immediate consequence of the conditions satisfied by the distance between two points.

It will now be shown that the condition is sufficient. The definition of distance is derived from the formula

$$(p,q) = \limsup \frac{|\phi(p) - \phi(q)|}{1 + |\phi(p) - \phi(q)|},$$

where all functions ϕ of the given equally continuous family are to be considered. It is evident that the function (p, q) just defined has the properties of distance. It remains to consider the relation between distance and point of accumulation.

Suppose that a point p is a point of accumulation of a set E. Because of the equal continuity of the functions ϕ there is for every positive number e a neighborhood V of p on which the oscillation of each function is less than e. Therefore the distance of the point p from the set E is zero.

Conversely, suppose that every sphere $S(p, a)$ of center p and radius a in the metric space just defined contains a point q of a set E. Let V be any neighborhood of the point p in the given space (V). By the second condition of the theorem there is a function ϕ and a number e such that $|\phi|>e$ on the set $P-V$. This implies that $(p, q)>e/2$ for all points q of $P-V$. Therefore the sphere $S(p, e/2)$ is contained in the neighborhood V. That is, V contains a point of E, and the point p is by definition a point of accumulation of E.

12. *The Equivalence of Distance and Uniformily Regular Écart.* Attempts have been made by Fréchet,* E. R. Hedrick,† A. D. Pitcher, and the writer‡ to obtain effective generalizations of the theory of metric spaces. That is, to impose hypotheses which yield substantially the same group of theorems about point sets and are less restrictive. It has however been established in each case that the conditions proposed imply that the resulting space is equivalent to a metric space.

In the theory proposed by Fréchet, distance is replaced by a function (p, q) with the properties (0), (1) of distance and the further property

(2′) *There is a function $f(e)$, approaching zero with e, such that for any three points p, q, r,*

$$(p,r) \leqq e, \qquad (q,r) \leqq e, \qquad \textit{imply} \quad (p,q) \leqq f(e).$$

* *Sur quelques points du calcul fonctionnel,* Rendiconti di Palermo, vol 30 (1906), p. 1–74.

† Transactions of this Society, vol. 12 (1911), pp. 285–294.

‡ Transactions of this Society, vol. 19 (1918), pp. 66–78.

The discovery by H. Hahn* that every space of this type which contains two or more distinct points admits continuous non-constant functions led Fréchet† to the conclusion that this class of spaces is metrizable. This conclusion was verified by Chittenden.‡

From this and other considerations Fréchet§ has in his later papers employed the term écart to refer to a non-negative single-valued real-valued function of two points. In this terminology the function (p, q) which was formerly called a voisinage becomes a uniformly regular écart.

THEOREM. *The concepts uniformly regular écart and distance are equivalent.*

Since distance is a uniformly regular écart for which $f(e) = 2e$ it is sufficient to show that a class of equally continuous functions satisfying the conditions of the theorem of §11 above may be defined in any space admitting a definition of its points of accumulation in terms of a uniformly regular écart.

If the given space P is singular the theorem is obvious. If it contains at least two points a number $a > 0$ can be chosen with the property: for every point p there is a point q such that $(p, q) > a$. This number a may be determined in the following manner. Let q', q'' be any two distinct points and let a be chosen so that $f(a) < (q', q'')$. Then one of the points q', q'' is effective as the required point q. For if $(p, q') \leqq a$, and $(p, q'') \leqq a$, then by the property (2′)

$$(q', q'') \leqq f(a),$$

contrary to the definition of a.

Suppose $a' < a$ chosen so that $f(a') < a$. Let p_0 be any fixed point and let||

$$A = S(p_0, a'), \quad B = P - S(p_0, a), \quad C = P - A - B.$$

* Monatshefte, vol. 19 (1908), pp. 247–257.

† Rendiconti di Palermo, vol. 30 (1910), p. 22–23.

‡ Transactions of this Society, vol. 18 (1917), pp. 161–166.

§ See Bulletin des Sciences Mathématiques, (2), vol. 42 (1918), pp. 1–19.

|| The validity of the following discussion is unimpaired in case the set C or any of its subdivisions is a null set.

The sets A^0, B^0 are disjoined. Suppose the contrary. Then there would exist points p, q, r of A, B, C, respectively, such that $(p, r) < a''$, $(q, r) < a''$, where $f(a'') < a'$. Then since $(p_0, p) < a'$, and $(p, q) \leqq f(a'') < a'$, we have $(p_0, q) \leqq a$, a contradiction.

The set C will be divided into disjoined sets C_0, C_1 according to the following rule. If $(r, A) = (r, B)$, where (r, A) denotes the distance from the point r to the set A, the point r is assigned to C_0, otherwise to C_1. The sets A^0, C_1^0 are disjoined, likewise the sets C_0^0, B^0. It is sufficient to give the proof for the first case. Let a number c be chosen so that $f(c) < a''$, and assume that there is a point r common to A^0, C_1^0. Then points p and q of A and C_1 respectively exist such that $(p, r) < c$, $(q, r) < c$. Therefore $(p, q) \leqq f(c) < a''$. From the definition of C_1 there is a point q' of B such that $(q, q') < (p, q) < a''$. Therefore $(p, q') \leqq f(a'') < a'$ contradicting the definition of the set B.

From the sets A, B, C we obtain by iteration of this method of subdivision a development of the space P of which the mth stage has the form

$$A, \; C_{00\cdots 0}, \; C_{00\cdots 01}, \; C_{i_1 i_2 \cdots i_m}, \; \cdots, \; C_{11\cdots 1}, \; B,$$

in which the indices i assume the values 0, 1 only. This sequence has the further property that there exists a number a_m (which is independent of p_0) such that the distance of any two non adjacent sets exceeds a_m. It is quite easily shown by mathematical induction that the numbers a_m may be so chosen that

$$a > a_1 > a_2 \cdots, \quad La_m = 0,$$

and that

$$a > f(a_1), \; a_1 > f(a_2), \; \cdots.$$

For each point p of C there is a unique set $C_{i_1 i_2 \ldots i_m}$ of stage m of which it is an element. Consequently each point determines a unique sequence of indices,

$$i_1, \; i_2, \; i_3, \; \cdots, \; i_m, \; \cdots,$$

and therefore corresponds to the number in the binary scale determined by these indices.

We proceed to the definition of a function ϕ of primary importance. On the set A, $\phi(p)=0$, on B, $\phi(p)=1$, on C

$$\phi(p) = \frac{i_1}{1} + \frac{i_2}{2^2} + \frac{i_3}{3^3} + \cdots .$$

This function is continuous. In fact, if $(p, q) < a_m$ then p and q lie in adjacent classes of the mth stage of the development of the space P and therefore

$$|\phi(p) - \phi(q)| \leqq 1/2^{m-1}.$$

Since the function ϕ is defined in terms of a and p_0, and the numbers a_m are independent of p_0, it follows that the family of all such functions obtained by varying p_0 and keeping a fixed is equally continuous.

If we now make the definitions

$$A_n = S(p_0, a_n), \qquad B_n = P - S(p_0, a_{n-1}),$$

$n=1, 2, 3, \cdots$, $a_0=a$, and let ϕ_n denote the function defined relative to A_n, B_n by the foregoing process, we obtain the desired family of equally continuous functions by considering all possible functions of the form

$$\phi = \sum_{n=1}^{\infty} \phi_n/2^n.$$

Since $\phi_n=1$ on B_n for each value of n, it follows that $\phi=1/2^n>0$ on the set $P-S(p_0, a_{n-1})$. Since $La_n=0$ the second condition of the theorem of § 11 is satisfied.

13. *Coherent Spaces.* Another attempt to generalize effectively the theory of metric spaces was made by A. D. Pitcher and E. W. Chittenden.* They considered an écart (p, q) in which the second condition on distance is replaced by the condition:

(2′) if $L(p,p_n)=0$, and $L(p_n,q_n)=0$, then $L(p,q_n)=0$.

A space in which the points of accumulation are definable in terms of a symmetric écart satisfying the condition (2′) is

* Loc. cit.

said to be coherent. It has recently been shown by Niemytski* that if the écart satisfies the further condition: $(p, q)=0$ is equivalent to $p=q$, then the space is metric.

14. *Spaces Defined by Developments.* The metrization problem for spaces whose points of accumulation are defined in terms of a development Δ was studied by Pitcher and Chittenden.† A development Δ is an arbitrary system of subclasses P^{ml} $(m=1, 2, 3, \cdots, l=1, 2, \cdots, l_m)$ of a fundamental class P in which the classes P^{ml} for a fixed index m form a stage Δ^m of the development.‡ The index l_m may have a finite or infinite range. If its range is finite for all values of m the development is said to be finite, otherwise infinite.

A point p is a point of accumulation of a set of points E relative to a development Δ provided there is a sequence of indices $m_1<m_2<m_3 \cdots$ and a corresponding sequence of classes $P^{m_n l_n}$ each of which contains an element of the set $E-p$. The reader is referred to the original article for the details of this investigation. The metrization problem was solved for compact spaces. The paper contains a set of necessary and sufficient conditions that a compact Hausdorff space be metric.

15. *The General Metrization Problem.* The general problem, to determine the topological conditions for the metrization of a topological space, was first explicitly stated and solved by Alexandroff and Urysohn.§ It is however of interest to note that E. R. Hedrick|| in continuing the search begun by Fréchet for a generalization of metric space discovered that a number of important theorems stated by Fréchet for classes

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* In an article to appear in the Transactions of this Society.

† Transactions of this Society, vol. 20 (1919), pp. 213–233.

‡ This definition is due to E. H. Moore, New Haven Mathematical Colloquium, Yale University Press, New Haven, 1910, pp. 1–150.

§ Comptes Rendus, vol. 177 (1923), p. 1274.

|| Loc. cit.

(V) normales* could be proved in any class (L)† in which derived sets are closed, providing the given space admits a property called the *enclosable* property. While Fréchet‡ soon proved that the space thus defined by Hedrick was in fact a "classe (V) normale" (therefore metric), it is important to observe that a slight modification of the conditions imposed by Hedrick constitute a set of necessary and sufficient conditions for the metrization of an abstract set.

We shall give three solutions of the general metrization problem; the first contains a modified form of the enclosable property of Hedrick, the second is due to Alexandroff and Urysohn, and the third is based upon the notion of coherence introduced by Pitcher and Chittenden. These three sets of conditions are alike in requiring the existence of a type of development Δ of fundamental importance which it is proposed to call *regular.*

Each stage of a *regular* development of a topological space is a family Δ^m of open sets V^m which covers the space P. The development proceeds by consecutive stages, that is, each set V^{m+1} of the $(m+1)$st stage is a subset of a set of the mth stage. Furthermore, if

$$V^1, V^2, V^3, \cdots, V^m, \cdots$$

is any infinite sequence of open sets one from each stage of the development and if there is a point p which is common to the sets of the sequence, then that point is *determined* by the sequence. That is, if V is any neighborhood of the point p, then for some value of the integer m, we have $V^m = V$. The sets V^m of the mth stage of the development are said to be of *rank m.*

The following additional definitions will be needed. Two points p, q are *developed* of stage or rank m provided there is a set of rank m which contains them both. In a regular develop-

* A "classe (V) normale" is a class that is separable, perfect, admits a definition of voisinage and a generalization of the theorem of Cauchy.

† A class (L) is essentially a space in which point of accumulation is defined in terms of the relation limit of a sequence.

‡ Transactions of this Society, vol. 14 (1913), pp. 320–324.

ment two points which are developed of rank m are developed of any lower rank.

Two sequences p_m, q_m are connected by a regular development provided the points p_m, q_m are developed of stage $m(m=1, 2, 3, \cdots)$.

A regular development Δ will be said to be *coherent* provided the connection of sequences is transitive. That is, a sequence $\{p_m\}$ is connected with a sequence $\{q_m\}$ whenever there is a sequence $\{r_m\}$ such that $\{p_m\}$ is connected with V_m and $\{r_m\}$ is connected with $\{q_m\}$.

The $(m+1)$st stage of a development is said to be *inscribed* in the mth if every pair of sets of rank $m+1$ which have a common point is contained in a set of rank m.

THEOREM. *Let P be a Hausdorff space admitting a regular development Δ in open sets V^m. Each of the following three conditions is a necessary and sufficient condition that P be equivalent to a metric space.*

I. (Hedrick) *For any positive integer m there is an integer n such that for any point p there is a set V^m of rank m which includes all sets of rank n which contain p.*

II. (Alexandroff and Urysohn) *For each value of the integer m the $(m+1)$st stage of the development Δ is inscribed in the m th.*

III. *The development Δ is coherent.*

Since it is evident that the three conditions of the theorem are necessary we may proceed at once to the proofs of their sufficiency.

Let us consider first the condition of Hedrick. It will be convenient to assume that the class P is a set of zero rank and allow the index m to have the values $m=0, 1, 2, 3, \cdots$. Then from condition II there exists for each integer n an integer $m=g(n)$, the greatest value of m for which n is the integer determined by condition II. The function $g(n)$ is unbounded. Suppose the contrary. Then there would exist an integer m' such that for every value of n, $g(n)<m'$. But there is an inte-

ger n' determined by m', contrary to the definition of $g(n)$ as the greatest such integer.

The écart (p, q) of two points is defined as follows. We set $(p, p)=0$. If m is the largest integer for which p and q are developed of rank m then $(p, q)=1/2^m$. Evidently the écart is symmetric. And if $p\neq q$ then $(p, q)>0$. For if the points p and q are developed of every rank m, then there is a sequence of sets V^m, one from each stage of the development to which both are common, contrary to the definition of a regular development, and the fourth condition of Hausdorff.

It remains to be shown that the écart thus defined is uniformly regular and that it defines the space P.

Let n be any integer and let p, q, r be three points such that $(p, r)<1/2^n$, $(q, r)<1/2^n$. If $m=g(n)$ we have at once, from condition II, that p and q are developed of stage m and therefore $(p, q)<1/2^m$. If $1/2^{n-1}<e\leqq 1/2^n$, and we set $f(e)=1/2^m$ we obtain a function $f(e)$ satisfying the required condition (2′).

It remains to show that the given space and the derived space are equivalent. It is evident that if every set V^m which contains p contains a point of a set E then the écart (p, E) is null. Conversely, if $(p, E)=0$ there is for every integer n a point q_n of E such that $(p, q_n)<1/2^n$. That is, p and q_n are common to some neighborhood V^n. It follows that q_n is an element of $V^{g(m)}(p)$. Since $Lg(n)=\infty$, it follows from the regularity of the development Δ that every neighborhood of p contains a point of E.

It will now be shown that condition II implies condition I. We assume a regular development Δ satisfying condition II. A regular development Δ_1 satisfying condition I for $n=m+1$ may be defined in terms of Δ as follows. Stage $\Delta_1{}^m$ of Δ_1 is the family of all open sets $U^m(p)$ obtained by forming the sum for each point p of all the sets V^m of rank m of Δ which contain p. To show that if a point p is common to a sequence of sets

$$U^1, U^2, U^3 \cdots, U^m, \cdots$$

then p is determined by that sequence, let V be any neighborhood of p, and suppose that for each value of m there is a point

q_m of U^m which does not belong to V. Since q_m is in the set U^m there must be a set V^m of rank m of which contains both p and q_m. But the sequence of sets V^m determines p. Therefore for some value of m, V^m (and therefore q_m) is contained in V, contrary to the hypothesis on q_m.

In the proof of the sufficiency of condition I the écart (p, q) of two points was defined, and without reference to that condition. This definition will be applied in the proof of the sufficiency of condition III. Because of the result obtained by Niemytski stated in § 13, we have only to show that this écart satisfies the condition (2″), since the equivalence of the spaces has already been established.

It is at once evident from the definition of écart that if two sequences are connected by the development then $L(p_n, q_n)=0$ and conversely. If then, $L(p, p_n)=0$, and $L(p_n, q_n)=0$, the sequence composed of the repeated element p must because of condition III be connected with the sequence q_n. Therefore $L(p, q)=0$, as required by condition (2″).

16. *The Borel Theorem.* A set E will be said to possess the "any-to-finite" form of the property of Borel in case every infinite proper covering F of E contains a finite proper covering. A family F of sets G is a proper covering of a set E if every element of E is interior to some set E. In a metric space a necessary and sufficient condition that a set E admit this property is that E be compact and closed. Thus the problem, to characterize those spaces in which the "any-to-finite" form of the property of Borel is equivalent to compact (or self-compact) and closed, is very closely related to the metrization problem. As Hildebrandt* has presented an excellent and full account of recent work on this problem, it is unnecessary to discuss it further here.

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* Loc. cit.