and conditions (3a), (3b) become

$$(7) \qquad A+D<B+C, \qquad AD=0.$$

Hence we find that *the most general existent boolean relation R which satisfies* (1) *is given by* (2) *and* (7).

Our main results may also be stated in the following form: *The totality of transitive universal relations in a boolean algebra is given by* (4). *The totality of existent transitive universal relations is given by*

$$Axy+Bxy'+Cx'y+Dx'y'=0, \quad A+D<B+C, \quad AD=0.$$

THE UNIVERSITY OF CALIFORNIA

ON CERTAIN QUINARY QUADRATIC FORMS*

BY E. T. BELL

1. *Introduction.* Except the classical theorems on the total number $N_5(n)$ of representations of the integer n as a sum of five integer squares, no explicit results on numbers of representations in quinary quadratic forms seem to have been obtained.† In general $N_5(n)$ is not expressible as a function of the real divisors alone of a single integer, but when n is a square, $N_5(n)$ is so expressible. This remarkable fact was found inductively by Stieltjes for $N_5(p^2)$, p prime, and proved for $N_5(n^2)$ by Hurwitz,‡ who showed that if $n=2^\alpha m$, m odd, $N_5(n^2)=10\zeta_3(2^\alpha)H(m)$, where

$$H(m)=[\zeta_3(p^a)-p\zeta_3(p^{a-1})][\zeta_3(q^b)-q\zeta_3(q^{b-1})]\ldots,$$

$\zeta_r(n)$ being the sum of the rth powers of all the divisors of n, and $m=p^a q^b\ldots$ the prime factor resolution of m; by convention $H(1)=1$. In the course of his proof he showed that

$$\zeta_1(m^2)+2\zeta_1(m^2-2^2)+2\zeta_1(m^2-4^2)+\cdots=H(m).$$

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† Cf. Bachmann, *Zahlentheorie*, vol. 4, pp. 565-594.

‡ COMPTES RENDUS, vol. 98 (1884), pp. 504-7; cf. Dickson's *History of the Theory of Numbers*, vol. 2, p. 311. For quadratic forms in $n>4$ variables, cf. ibid., vol. 3, chap. XI.

This identity makes it possible to find numerous theorems for the total number $N(2^{\alpha}m^2)$ of representations of $2^{\alpha}m^2$ in a quinary quadratic form other than a sum of five squares similar to Hurwitz' for $N_5(2^{2\alpha}m^2)$, viz., $N(n)$ is a function of the real divisors alone of a single integer when and only when* $n = 2^{\alpha}m^2$, m odd, $\alpha \geqq 0$, and similarly for the number of proper representations. The results are so unexpectedly simple that we shall take space in § 3 to give 19 of them, or, counting the separate cases according to α, 72.

2. *Theorems on Representations.* Henceforth the variables x, y, z, t, u may take any integer values $\gtreqless 0$; m, n, α, k_r are constant positive integers, m is odd, $n, \alpha \geqq 0$ are arbitrary, $k_r = 0$ when $r < 0$, and a, b, c, d, e are constant integers $\lessgtr 0$.

THEOREM I. *If the total number of representations of* $2^{\alpha}m$ *in the form* $ax^2 + by^2 + cz^2 + dt^2$ *is* $k_{\alpha}\zeta_1(m)$, *then the total number of representations of* $2^{\alpha}m^2$ *in the form* $(a, b, c, d, \alpha) \equiv ax^2 + by^2 + cz^2 + dt^2 + 2^{\alpha+2}u^2$ *is* $k_{\alpha}H(m)$.

Let $N'(n)$, $N(n)$ denote respectively the total numbers of representations of n in the forms $ax^2 + by^2 + cz^2 + dt^2$, $ax^2 + by^2 + cz^2 + dt^2 + eu^2$. Then, the sum continuing so long as the argument of N' is positive, evidently

$$N(n) = N'(n) + 2N'(n - e1^2) + 2N'(n - e2^2) + \cdots.$$

If $n = 2^{\alpha}m^2$, $e = 2^{\alpha+2}$, the argument of each N' is an odd multiple of 2^{α}. Hence

$$N(2^{\alpha}m^2) = k_{\alpha}[\zeta_1(m^2) + 2\zeta_1(m^2 - 2^2) + 2\zeta_1(m^2 - 4^2) + \cdots],$$

and the theorem follows by Hurwitz' identity.

As customary, a representation, in which the G.C.D. of x, y, z, t, u is 1 is called proper. To state the number of proper representations when the total number is given by Theorem I, we need $J_2(m)$, the Jordan totient of order 2 of m. If $m = \prod p_{\alpha}$ is the resolution of m into powers of distinct primes >1, $J_2(m) = m^2 \prod (1 - p^{-2})$, $J(1) = 1$ by convention, and $J_2(m)$ is the number of sets of two equal or distinct

* Because in each case it can be shown that $N(n) = cN_5(n)$, where c is a constant.

positive integers $\leqq m$ whose G.C.D. is prime to m. Applying the general method of a former paper* to Theorem I, we find readily the following result.

THEOREM II. *If the total number of representations of* $2^\alpha m^2$ *in the form* (a, b, c, d, α) *is* $k_\alpha H(m)$, *the number of proper representations of* $2^\alpha m^2$ *in the same form is* $(k_\alpha - k_{\alpha-2})m\ J_2(m)$.

It is obvious that precisely similar considerations can be applied to any quadratic form in any number of variables, also to such forms when certain of the variables are restricted (examples in § 3 (2), (5)). The discussion is limited here to (a, b, c, d, α) because the application of the method to the deduction of explicit results for other forms presupposes a study of the general class of identities of which Hurwitz' is the second simplest example. The first such identity concerns the function which gives the number of representations of an integer as a sum of two squares. In both this and Hurwitz' identity only odd divisors enter; in the general case there appear only those divisors which are prime to the prime $p > 2$.

3. *Numbers of Representations.* From the way in which Theorems I, II are stated it is sufficient for any given form (a, b, c, d, α) to record the values of k_α and $k'_\alpha \equiv k_\alpha - k_{\alpha-2}$. The values of k_α are those which appertain to the quaternary form obtained from (a, b, c, d, α) by omitting the term in u. They can be found by referring to the papers of Liouville and Pepin.†

The results for quinary forms are given in the following list, in which the asterisk (*) denotes any k with a suffix greater than that of the last k in the symbol in which the asterisk occurs. Thus (11) states that the total number

* ANNALS OF MATHEMATICS, (2), vol. 21 (1920), p. 170, § 8.

† JOURNAL DES MATHÉMATIQUES, (2), vol. 6 (1861), pp. 409, 440; vol. 7 (1862), pp. 1, 76, 80, 107, 112, 115, 119, 154, 156, 160, 163, 167; (4), vol. 6 (1890), pp. 5–67.

of representations of $2^{\alpha}m^2$ (m odd) in the form

$$x^2+2y^2+2z^2+4t^2+2^{\alpha+2}u^2$$

is 2, 4, 8, or 24 times $H(m)$ according as

$$\alpha=0,\ 1,\ 2\ \text{ or } > 2.$$

(1) $(1,1,1,4,\alpha):[k_0,k_1,k_2,*]=[6,12,8,24].$
(2) $(1,1,1,4,\alpha),\ x+y \text{ odd}: k_1=8.$
(3) $(1,1,1,16,\alpha):[k_2,k_3,k_4,*]=[6,12,8,24].$
(4) $(1,1,2,2,\alpha):[k_0,k_1,*]=[4,8,24].$
(5) $(1,1,2,2,\alpha),\ xy \text{ odd}: [k_1,k_2,*]=[4,16,0].$
(6) $(1,1,2,8,\alpha):[k_1,k_2,k_3,*]=[6,12,8,24].$
(7) $(1,1,4,4,\alpha):[k_0,k_1,k_2,*]=[4,4,8,24].$
(8) $(1,1,4,16,\alpha):[k_2,k_3,k_4,*]=[6,12,8,24].$
(9) $(1,1,8,8,\alpha):[k_1,k_2,k_3,*]=[4,4,8,24].$
(10) $(1,1,16,16,\alpha):[k_2,k_3,k_4,*]=[4,4,8,24].$
(11) $(1,2,2,4,\alpha):[k_0,k_1,k_2,*]=[2,4,8,24].$
(12) $(1,2,2,16,\alpha):[k_2,k_3,k_4,*]=[6,12,8,24].$
(13) $(1,2,4,8,\alpha):[k_1,k_2,k_3,*]=[2,4,8,24].$
(14) $(1,2,8,16,\alpha):[k_2,k_3,k_4,*]=[2,4,8,24].$
(15) $(1,4,4,4,\alpha):[k_0,k_1,k_2,*]=[2,0,8,24].$
(16) $(1,4,4,16,\alpha):[k_2,k_3,k_4,*]=[6,12,8,24].$
(17) $(1,4,8,8,\alpha):[k_1,k_2,k_3,*]=[0,4,8,24].$
(18) $(1,4,16,16,\alpha):[k_1,k_2,k_3,k_4,*]=[0,4,4,8,24].$
(19) $(1,16,16,16,\alpha):[k_2,k_3,k_4,*]=[2,0,8,24].$

When combined with the theorems on the number of representations of an integer as a sum of two and of three squares, all of these give interesting binary quadratic class number relations of a new kind, examples of which are presented in another paper (not yet published). To save space we shall omit the numbers of proper representations, which can be written down from the above by Theorem II; e. g., for the form (1) we get

$$(k_0', k_1', k_2', k_3', k_4', *)=(6,12,2,12,16,0).$$

THE UNIVERSITY OF WASHINGTON