CONDITION THAT A TENSOR BE THE CURL OF A VECTOR *

BY L. P. EISENHART

It is the purpose of this note to establish the following theorem.

THEOREM. *A necessary and sufficient condition that a covariant skew-symmetric tensor A_{ij} in a space of any order n be expressible in terms of n functions φ_i in the form*

$$A_{ij} = \frac{\partial \varphi_i}{\partial x^j} - \frac{\partial \varphi_j}{\partial x^i} \tag{1}$$

is that

$$\frac{\partial A_{ij}}{\partial x^k} + \frac{\partial A_{jk}}{\partial x^i} + \frac{\partial A_{ki}}{\partial x^j} = 0, \quad (i, j, k = 1, \cdots, n). \tag{2}$$

Consider first the case of 3-space. If φ_2 and φ_3 are any two functions such that

$$A_{23} = \frac{\partial \varphi_2}{\partial x^3} - \frac{\partial \varphi_3}{\partial x^2},$$

the conditions of integrability of

$$\frac{\partial \varphi_1}{\partial x^2} = \frac{\partial \varphi_2}{\partial x^1} + A_{12}, \qquad \frac{\partial \varphi_1}{\partial x^3} = \frac{\partial \varphi_3}{\partial x^1} + A_{13}$$

are satisfied in consequence of (2), and the theorem is established for 3-space.

Now we show that, if the theorem is true for n-space, it is true for $(n + 1)$-space. On this assumption equations (1) hold for $i, j = 1, \cdots, n$. For a particular i and j and for $k = n + 1$, equation (2) may be written in the form

$$\frac{\partial}{\partial x^i}\left(A_{jn+1} - \frac{\partial \varphi_j}{\partial x^{n+1}}\right) = \frac{\partial}{\partial x^j}\left(A_{in+1} - \frac{\partial \varphi_i}{\partial x^{n+1}}\right).$$

Hence a function φ_{n+1} is defined by the equations

$$A_{in+1} = \frac{\partial \varphi_i}{\partial x^{n+1}} - \frac{\partial \varphi_{n+1}}{\partial x^i}, \qquad A_{jn+1} = \frac{\partial \varphi_j}{\partial x^{n+1}} - \frac{\partial \varphi_{n+1}}{\partial x^j}. \tag{3}$$

* Presented to the Society, September 7, 1922.

Replacing j in (2) by l ($= 1, \cdots, n;\ \neq j$), we have, by (3),

$$\frac{\partial A_{l\,n+1}}{\partial x^i} = \frac{\partial}{\partial x^i}\left(\frac{\partial \varphi_l}{\partial x^{n+1}} - \frac{\partial \varphi_{n+1}}{\partial x^l}\right).$$

Consequently (1) holds for $i, j = 1, \cdots, n+1$, and the theorem is established. It should be remarked that one of the functions φ_i may be chosen arbitrarily, or what is equivalent, that the functions φ_i are determined to within additive functions $\partial\psi/\partial x^i$, where ψ is an arbitrary function of the x's.

Thus far we have made no use of the fact that A_{ij} are the components of a covariant tensor. If $A'_{\alpha\beta}$ denote the components of the tensor in terms of coordinates x', then

$$A'_{\alpha\beta} = A_{ij}\frac{\partial x^i}{\partial x'^\alpha}\frac{\partial x^j}{\partial x'^\beta}. \tag{4}$$

If Γ^i_{jk} and $\Gamma'^\alpha_{\beta\gamma}$ denote the Christoffel symbols of the second kind for the respective systems of coordinates x and x' of a Riemannian geometry, then*

$$\frac{\partial^2 x^p}{\partial x'^i \partial x'^j} = \Gamma'^t_{ij}\frac{\partial x^p}{\partial x'^t} - \Gamma^p_{qr}\frac{\partial x^q}{\partial x'^i}\frac{\partial x^r}{\partial x'^j}.$$

The same equations obtain in the more general case of a geometry of paths, where the functions $\Gamma^i_{\alpha\beta}$ and $\Gamma'^\alpha_{\beta\gamma}$ are the coefficients of the equations of the paths in the two systems of coordinates.† By means of these equations we show that, if the functions A_{ij} satisfy (2), so also do $A'_{\beta\gamma}$ defined by (4). In consequence of the above theorem equation (4) may be replaced by the equation

$$\frac{\partial \varphi_\alpha'}{\partial x'^\beta} - \frac{\partial \varphi_\beta'}{\partial x'^\alpha} = \left(\frac{\partial \varphi_i}{\partial x^j} - \frac{\partial \varphi_j}{\partial x^i}\right)\frac{\partial x^i}{\partial x'^\alpha}\frac{\partial x^j}{\partial x'^\beta} = \frac{\partial \varphi_i}{\partial x'^\beta}\frac{\partial x^i}{\partial x'^\alpha} - \frac{\partial \varphi_j}{\partial x'^\alpha}\frac{\partial x^j}{\partial x'^\beta}$$
$$= \frac{\partial}{\partial x'^\beta}\left(\varphi_i\frac{\partial x^i}{\partial x'^\alpha}\right) - \frac{\partial}{\partial x'^\alpha}\left(\varphi_j\frac{\partial x^j}{\partial x'^\beta}\right).$$

Hence

$$\varphi_\alpha' = \varphi_i\frac{\partial x^i}{\partial x'^\alpha} + \frac{\partial \psi}{\partial x'^\alpha}, \tag{5}$$

* Bianchi, *Lezioni*, vol. 1, p. 64.

† See Proceedings of the National Academy, vol. 8 (1922), p. 21.

where ψ is an arbitrary function.

From (5) it is evident that if A_{ij} are defined as the components of the curl of covariant vector, then (2) are necessarily satisfied; but (2) is not a sufficient condition. That this condition is not sufficient was overlooked by me in a recent paper,* and my conclusions in § 5 are not correct. In fact, the skew-symmetric tensor there defined by S_{ij} is given by

$$S_{ij} = \frac{\partial \Gamma^{\alpha}_{\alpha j}}{\partial x^{i}} - \frac{\partial \Gamma^{\alpha}_{\alpha i}}{\partial x^{j}},$$

and the functions $\Gamma^{\alpha}_{\alpha i}$ and $\Gamma'^{\alpha}_{\alpha i}$ in two sets of coordinates are in the relation

$$\Gamma'^{\alpha}_{\alpha i} = \Gamma^{\alpha}_{\alpha j} \frac{\partial x^{j}}{\partial x'^{i}} + \frac{\partial}{\partial x'^{i}} \log \Delta,$$

where Δ is the Jacobian of the transformation.

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A NEW GENERALIZATION OF TCHEBYCHEFF'S STATISTICAL INEQUALITY

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1. *Introduction.* If $f(x)$ is any frequency distribution, and s its standard deviation, the symbol $P(\lambda s)$ may be used to represent the probability that a datum drawn from this distribution will differ from the mean value by as much as λs, numerically. For the solution of various statistical problems it is desirable to have a formula which will measure $P(\lambda s)$ when $f(x)$ is only partially known. A case of practical importance occurs when $f(x)$ represents the distribution of values of a statistical constant determined by sampling from a known distribution, such a constant as, for example, a mean value, or a coefficient of correlation. In such cases it is usually difficult or impossible to find the complete distribution $f(x)$, but quite feasible to find its lower moments. Tchebycheff's well known inequality is: $P(\lambda s) \leqq 1/\lambda^2$. It has been general-

* PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 8 (1922), p. 236.