CIRCLE AND SPHERE GEOMETRY.

A Treatise on the Circle and the Sphere. By JULIAN LOWELL COOLIDGE. Oxford, the Clarendon Press, 1916. 8vo. 603 pp.

AN era of systematizing and indexing produces naturally a desire for brevity. The excess of this desire is a disease, and its remedy must be treatises written, without diffuseness indeed, but with the generous temper which will find room and give proper exposition to true works of art. The circle and the sphere have furnished material for hundreds of students and writers, but their work, even the most valuable, has been hitherto too scattered for appraisement by the average student. Cyclopedias, while indispensable, are tantalizing. Circles and spheres have now been rescued from a state of dispersion by Dr. Coolidge's comprehensive lectures. Not a list of results, but a well digested account of theories and methods, sufficiently condensed by judicious choice of position and sequence, is what he has given us for leisurely study and enjoyment.

The reader whom the author has in mind might be, well enough, at the outset a college sophomore, or a teacher who has read somewhat beyond the geometry expected for entrance. Beyond the sixth chapter, however, or roughly the middle of the book, he must be ready for persevering study in specialized fields. The part most read is therefore likely to be the first four chapters, and a short survey of these is what we shall venture here.

After three pages of definitions and conventions the author calls "A truce to these preliminaries!" and plunges into inversion. Circles orthogonal to a fixed circle are self-inverse, and this property generalized defines anallagmatic curves. A family of such circles will have an anallagmatic envelope, for example. A novel term, deferrent, means the locus of centers of a family of orthogonal circles. In 35 closely linked theorems we advance to mutually tangent circles, and find at once a demonstration of Steiner's favorite theorem on poristic systems of circles, whose concise formulation warrants quotation (page 34): "Given two non-intersecting circles which possess the property that a ring of n circles may be constructed all tangent to them and successively tangent to one another making m complete circuits, and if two circles of the ring touch the original ones at points on one circle orthogonal to these two, then the original circles are members of a ring of n_1 circles making m_1 complete circuits, all tangent to the two of the first ring, where

$$\frac{m}{n}+\frac{m_1}{n_1}=\frac{1}{2}."$$

Proving soon after this a condition given by Casey for four mutually external circles tangent to a fifth, the author reduces the four to points upon the fifth circumference and so derives Ptolemy's theorem: "If a convex quadrilateral be inscribed in a circle, the sum of the products of the opposite sides is equal to the product of the diagonals," these lengths being the segments on common tangents employed in Casey's formula. The converses of both are established; then follows the nine-point circle.

A section on circles related to a triangle brings in the names of Euler, Simson, Barrow, Fontené, Nagel, Fuhrmann, Greiner, and Kantor. We cite one theorem whose lowest case is commonly given in textbooks (page 48). "If a polygon be inscribed in a circle and tangents be drawn at all of its vertices, the product of the distances of any point of the circle from these tangents is equal to the product of its distances from the side-lines."

Brocard points of a triangle, Lemoine's circles, Steiner's point, Tarry's point, and others, fill a twenty-page section on the geometry of the triangle (not so entitled, the author restricting his selection severely). Concurrent circles and concyclic points then claim attention,—Miquel's circle, and others due to Grace, Pesci, and our author himself (see *Annals of Mathematics*, series 2, volume 12). Coaxal systems of circles (18 pages) close the first and properly the longest chapter, on the circle in elementary plane geometry. We quote two of the shortest theorems. "The three circles on the diagonals of a complete quadrilateral as diameters are coaxal." This is ascribed to Gauss and Bodenmiller, 1830. "If a circle so moves that it cuts two others in diametrically opposite points, or cuts one in diametrically opposite points and the other orthogonally, it will generate a coaxal system" (page 108).

"Suggestions" are appended to this and to later chapters, such as this: "It seems likely that there are other simple criteria for various systems of tangent circles like Casey's condition for four circles tangent to a fifth, Vahlen's criterion for poristic systems, or the Euler conditions that there may be a triangle or a quadrilateral inscribed in one circle which is circumscribed to the other. . . . It seems likely that there are other chains of concurrent circles and concyclic points besides those noticed in theorems 162–6," etc. Comments of this kind, following protracted expositions reaching often a high degree of intrinsic beauty, are eminently in place.

Chapter 2 treats of the circle in cartesian plane geometry. A labored section on the treatment of circles by trilinear coordinates bears out the conclusion that those coordinates "are not adapted to dealing with the circle in any broad way," but "only in studying those properties of a circle which are related to a particular triangle." Tetracyclic coordinates are therefore introduced, so defined that if (xy) denotes the bilinear sum $\Sigma_1^4 x_i y_i$, points or zero circles have $(xx) = 0$. A neat form of identical relation among ten circles, ascribed either to Darboux or to Frobenius, is then much employed in specialized applications. It consists in the vanishing of a five-rowed determinant, obtained by separating the ten circles into two sets of five, denoted by x, y, z, s, t and x', y', z', s', t', each letter indicating of course four tetracyclic coordinates. The constituents of the determinant are then the 25 bilinear sums like

$$(xx'), (xy'), \ldots, (tx'), (ty'), \ldots, (tt').$$

The examples given, 21 in number, are all decidedly interesting.

Of greater interest is doubtless the fourth section of this chapter, on analytical systems. Some of Morley's memorable work figures here. A less known line of discussion is that on a general cubic series of circles (page 159 seq.). "The common orthogonal circles to corresponding triads in three projective pencils of circles whereof no two have a common member will generate a general cubic series, and every general cubic series may be so generated in ∞ ways." Also: "The locus of the centers of the circles of a general cubic series is a rational curve of the third order." Obviously the subject is the direct homotype of the twisted cubic curve in point space of three dimensions; but as incidence in space is orthogonality between circles, this version is a possible source of fruitful suggestions. Congruences, or two-parameter families, receive due attention, and the field partly developed is recommended for further exploration.

Famous problems in construction, the third chapter, is unique in the full discussion it gives of each problem with respect to Lemoine's characters of simplicity and exactness. The chief problems are those of Apollonius (to construct a circle tangent to three given circles), of Steiner (to construct a circle meeting three given circles at given angles), Malfatti's (to construct three circles, each of which shall touch the other two, and two sides of a given triangle), and two of Fiedler's. The chapter closes with an account of Mascheroni's geometry of the compasses.

Of later chapters, on more special themes, we mention those on pentaspherical space, on circle transformations and sphere transformations, on the oriented circle (and Laguerre transformations); on algebraic systems of circles in space, and on oriented circles in space. In particular we commend to readers the "suggestions" (page 473) that close the chapter on circle crosses; and the paragraph (page 482) on the pentacycle of Stephanos, and that which follows it.

The book has excellent indices, and an added page of errata and addenda. As a whole, it is a welcome gift to American and English geometricians. It has in prospect a long career of usefulness as a compendium of accessible fact and as a source of stimulation and suggestion. Incidentally it affords the refreshing detached humor that marks it as authentic.

H. S. WHITE.

SHORTER NOTICES.

Unified Mathematics. By L. C. KARPINSKI, H. Y. BENEDICT and J. W. CALHOUN. New York, D. C. Heath and Company, 1918. viii + 522 pp.

THE increasing popularity of a unified course in mathematics for college freshmen is reflected in the large number of text books covering this field which have recently been published. Moreover, the movement towards the general adoption of