THE DERIVATIVE OF A FUNCTIONAL.

BY PROFESSOR P. J. DANIELL.

IN his book on Integral Equations* Volterra has given a definition of the derivative of a functional and has stated somewhat restricted conditions under which the variation can be expressed as a linear integral. In the present paper it is shown that, under more general conditions, the variation is a linear functional in the sense of Riesz* and, therefore, a Stieltjes integral. This theorem is assumed as a condition in a paper by Fréchet.† Let

$$F[f\overset{b}{\underset{a}{(x)}}]$$

denote a functional F of a continuous function $f(x)$ $(a \leqq x \leqq b)$. With Volterra we shall consider only continuous functions. Let us denote the first variation by

$$D(f\,;\,\varphi) = \lim_{\epsilon \doteq 0} \frac{1}{\epsilon}\,\{F[f + \epsilon\varphi] - F[f]\}.$$

In place of Volterra's four conditions we take the two following:

I. $F[f]$ satisfies the Cauchy-Lipschitz condition, namely that we can find a number M such that

$$|F[f_1] - F[f_2]| \leqq M \max|f_1(x) - f_2(x)|.$$

II. The first variation $D(f'\,;\,\varphi)$ exists for all continuous φ, and all continuous f' in the neighborhood of f; that is to say that a number $\eta > 0$ can be found so that the variation exists so long as

$$\max|f'(x) - f(x)| \leqq \eta.$$

Under these conditions the variation is a linear functional, and therefore a Stieltjes integral,

$$D(f\,;\,\varphi) = \int_a^b \varphi(x)d\alpha(x).$$

* V. Volterra, Equations Intégrales, p. 12 et seq. F. Riesz, *Annales de l'Ecole Normale Supérieure*, vol. 31 (1914), p. 9.

† M. Fréchet, *Transactions Amer. Math. Society*, vol. 15 (1914), p. 135.

In the first place

$$(1) \qquad \begin{aligned} D(f\,;\,l\varphi) &= \lim_{\epsilon \doteq 0} \frac{l}{l\epsilon} \{F[f + \epsilon l\varphi] - F[f]\} \\ &= lD(f\,;\,\varphi). \end{aligned}$$

If we choose $\epsilon > 0$ so small that

$$\epsilon|l|\max|\varphi_1| + \epsilon|m|\max|\varphi_2| < \eta,$$

$D(f + \epsilon m\varphi_2\,;\,\varphi_1)$ will exist by II, and

$$F[f + \epsilon l\varphi_1 + \epsilon m\varphi_2] - F[f + \epsilon m\varphi_2] = \epsilon lD(f + \epsilon m\varphi_2;\,\varphi_1) + \epsilon\delta,$$

$$F[f + \epsilon l\varphi_1] - F[f] = \epsilon lD(f\,;\,\varphi_1) + \epsilon\delta',$$

where δ, δ' approach 0 with ϵ. Then

$$(2) \qquad \begin{aligned} P(l\varphi_1, m\varphi_2) &= \lim_{\epsilon \doteq 0} \frac{1}{\epsilon} \{F[f + \epsilon l\varphi_1 + \epsilon m\varphi_2] - F[f + \epsilon m\varphi_2] \\ &\qquad - F[f + \epsilon l\varphi_1] + F[f]\} \\ &= l \lim_{\epsilon \doteq 0} [D(f + \epsilon m\varphi_2\,;\,\varphi_1) - D(f\,;\,\varphi_1)]. \end{aligned}$$

Similarly

$$(3) \qquad P(l\varphi_1, m\varphi_2) = m \lim_{\epsilon \doteq 0} [D(f + \epsilon l\varphi_1\,;\,\varphi_2) - D(f\,;\,\varphi_2)].$$

The expression (2) is the product of l and a function of m independent of l, while (3) is the product of m and a function of l only, and they are equal. Each must be a product of lm and an expression K independent of l, m.

$$P(l\varphi_1, m\varphi_2) = lmK(\varphi_1, \varphi_2).$$

In this make $l = 1 = m$; then

$$P(\varphi_1, \varphi_2) = K(\varphi_1, \varphi_2).$$

Or

$$P(l\varphi_1, m\varphi_2) = lmP(\varphi_1, \varphi_2).$$

Making $m = l$,

$$P(l\varphi_1, l\varphi_2) = l^2P(\varphi_1, \varphi_2).$$

But

$$\begin{aligned} P(l\varphi_1, l\varphi_2) &= l \lim_{\epsilon \doteq 0} \frac{1}{\epsilon l} \{F[f + \epsilon l\varphi_1 + \epsilon l\varphi_2] \\ &\qquad - F[f + \epsilon l\varphi_2] - F[f + \epsilon l\varphi_1] + F[f]\} \\ &= lP(\varphi_1, \varphi_2). \end{aligned}$$

$$\therefore \quad l^2P(\varphi_1, \varphi_2) = lP(\varphi_1, \varphi_2)$$

$$P(\varphi_1, \varphi_2) = 0.$$

Or

$$\lim_{\epsilon \doteq 0} \frac{1}{\epsilon} \{F[f + \epsilon\varphi_1 + \epsilon\varphi_2] - F[f] - F[f + \epsilon\varphi_2] + F[f] - F[f + \epsilon\varphi_1] + F[f]\} = 0,$$

$$D(f\,;\, \varphi_1 + \varphi_2) - D(f\,;\, \varphi_2) - D(f\,;\, \varphi_1) = 0. \tag{4}$$

Combining (1) and (4), we see that

$$D(f\,;\, c_1\varphi_1 + c_2\varphi_2) = c_1 D(f\,;\, \varphi_1) + c_2 D(f\,;\, \varphi_2).$$

Thus the variation is distributive in φ. Secondly, from condition I,

$$|F[f + \epsilon\varphi] - F[f]| \leqq M\epsilon \max|\varphi|,$$

or

$$|D(f;\, \varphi)| = \lim_{\epsilon \doteq 0} \frac{1}{\epsilon} |F[f + \epsilon\varphi] - F[f]| \leqq M \max|\varphi|.$$

The variation is also bounded, considered as an operation on φ. This proves it to be a linear functional by Riesz's definition. To find the integrating function $\alpha(x)$ we may proceed as follows:

Let $\varphi(x;\, c,\, d)$ denote the continuous function,

$$\begin{aligned} \varphi &= 1, \quad a \leqq x \leqq c, \\ &= 0, \quad d \leqq x \leqq b, \end{aligned}$$

φ linear from c to d.

Then

$$\alpha(c) = \lim_{d \doteq c} D(f;\, \varphi),$$

and in general for any continuous $\varphi(x)$,

$$D(f;\, \varphi) = \int_a^b \varphi(x)\, d\alpha(x).$$

RICE INSTITUTE,
HOUSTON, TEXAS.