APPLICATIONS OF THE THEORY OF SUMMABILITY TO DEVELOPMENTS IN ORTHOGONAL FUNCTIONS.

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THE very considerable body of literature which may be described by the above title belongs almost entirely to the present century. Its extent is only roughly indicated by the bibliography at the end of the paper, which makes no pretensions to being complete. The type of series considered here constitutes one of the three most important classes of series to which the theory of summability has been applied, the other two being power series and Dirichlet's series. A noteworthy feature of the applications with which we shall be concerned is found in their usefulness in an important branch of applied mathematics, namely the theory of the flow of heat and electricity.

§1. *The Summability of Fourier's Series.*

The first writer to deal with the topic of this section was Fejér. In his fundamental paper of 1903 [5]* he established among other results the summability $(C1)$ of the Fourier development of an arbitrary function satisfying very wide conditions, at all points where the function is continuous or has a finite jump. We shall give a proof of this theorem, under somewhat modified conditions, which is substantially the same as Fejér's proof.

The Fourier development of $f(x)$ may be written in the form

$$(1) \quad \frac{1}{2\pi} \int_a^{2\pi+a} f(\theta)d\theta + \frac{1}{\pi} \sum_{n=1}^{\infty} \int_a^{2\pi+a} f(\theta) \cos n(\theta - x)d\theta,$$

it being assumed that $f(x)$ is periodic of period 2π. We have for the sum of the first n terms of (1)

$$(2) \quad s_n(x) = \frac{1}{\pi} \int_a^{2\pi+a} f(\theta) \frac{\cos (n-1)(\theta - x) - \cos n(\theta - x)}{2(1 - \cos (\theta - x))} d\theta.$$

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* The numbers in brackets refer to the bibliographical list at the end of the paper.

Hence for the arithmetic mean of the first n sums, we have

$$(3) \qquad \frac{S_n(x)}{n} = \frac{1}{2n\pi} \int_a^{2\pi+a} f(\theta) \frac{\sin^2 \dfrac{n(\theta - x)}{2}}{\sin^2 \dfrac{\theta - x}{2}} \, d\theta.$$

If now we set $t = \frac{1}{2}(\theta - x)$ and then take $\alpha = x - \pi$, we obtain

$$(4) \qquad \frac{S_n(x)}{n} = \frac{1}{n\pi} \int_{-(\pi/2)}^{\pi/2} f(x + 2t) \frac{\sin^2 nt}{\sin^2 t} \, dt$$

$$= \frac{1}{n\pi} \int_0^{\pi/2} \{f(x + 2t) + f(x - 2t)\} \frac{\sin^2 nt}{\sin^2 t} \, dt.$$

We are now ready for the proof of Fejér's theorem. We begin by establishing two lemmas.

LEMMA 1. *For any positive integer n, we have*

$$(5) \qquad \frac{1}{n\pi} \int_0^{\pi/2} \frac{\sin^2 nt}{\sin^2 t} \, dt = \tfrac{1}{2}.$$

If we set

$$(6) \quad \sigma_n(2t) = \tfrac{1}{2} + \cos 2t + \cos 4t + \cdots + \cos 2(n - 1)t$$

$$= \frac{\cos 2(n - 1)t - \cos 2nt}{2(1 - \cos 2t)},$$

we have

$$(7) \quad \frac{\Sigma_n(2t)}{n} = \frac{\sigma_1(2t) + \sigma_2(2t) + \cdots + \sigma_n(2t)}{n} = \frac{1}{2n} \cdot \frac{\sin^2 nt}{\sin^2 t}.$$

But from (6)

$$\frac{2}{\pi} \int_0^{\pi/2} \sigma_n(2t) dt = \tfrac{1}{2} \qquad (n = 1, 2, 3, \cdots).$$

From this relationship and (7) the identity (5) readily follows.

LEMMA 2. *If $\varphi(t)$ is integrable (Lebesgue) in the interval $(0 \leqq x \leqq c \leqq \frac{1}{2}\pi)$ and furthermore $\lim\limits_{t \to 0} \varphi(t) = 0$, then*

$$\lim_{n \to \infty} \frac{1}{n\pi} \int_0^c \varphi(t) \frac{\sin^2 nt}{\sin^2 t} \, dt = 0.$$

Given an arbitrary positive ϵ, we chose $\delta < \epsilon$ and such that

(8) $|\varphi(t)| < \epsilon$ $(0 \leqq t \leqq \delta)$.

Then, δ being fixed, we choose m so large that

(9) $\dfrac{1}{n\pi \sin^2 \delta} \displaystyle\int_\delta^c |\varphi(t)| \, dt < \dfrac{\epsilon}{2}$ $(n \geqq m)$.

From (8), (5), and (9) we obtain

$$\left| \frac{1}{n\pi} \int_0^c \varphi(t) \frac{\sin^2 nt}{\sin^2 t} \, dt \right| \leqq \frac{1}{n\pi} \int_0^\delta |\varphi(t)| \frac{\sin^2 nt}{\sin^2 t} \, dt$$

$$+ \frac{1}{n\pi} \int_\delta^c |\varphi(t)| \frac{\sin^2 nt}{\sin^2 t} \, dt < \frac{\epsilon}{n\pi} \int_0^{\pi/2} \frac{\sin^2 nt}{\sin^2 t} \, dt$$

$$+ \frac{1}{n\pi \sin^2 \delta} \int_\delta^c |\varphi(t)| \, dt < \epsilon \qquad (n \geqq m),$$

which proves our lemma.

FEJÉR'S THEOREM.* *If the function $f(x)$ is periodic of period 2π and is integrable (Lebesgue) over any interval of length 2π, the series* (1) *will be summable* (C1) *to the value* $\frac{1}{2}\{f(x + 0) + f(x - 0)\}$ *at every point for which this limit exists.*

It is obvious that this theorem can be proved by showing that

$$\lim_{n \to \infty} \left[\frac{S_n(x)}{n} - \frac{1}{2}\{f(x + 0) + f(x - 0)\} \right] = 0,$$

where $S_n(x)$ is defined by (3). In view of (4) and Lemma 1, the expression in brackets may be written in the form

$$\frac{1}{n\pi} \int_0^{\pi/2} \{f(x + 2t) + f(x - 2t) - f(x + 0) - f(x - 0)\} \frac{\sin^2 nt}{\sin^2 t} \, dt.$$

It follows at once from Lemma 2 that this last expression approaches zero as n becomes infinite. The theorem is therefore proved.†

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* Fejér's original conditions are somewhat different from ours, he having required that $f(x)$ be integrable (Riemann) and become infinite at only a finite number of points. Thus his result and our result overlap. It is easy to modify our argument so as to include both results.

† With slight changes the above argument may be used to establish uniform summability throughout any interval included in an interval of continuity of $f(x)$.

Since the points of discontinuity of a function having a Riemann integral form a set of measure zero, Fejér's theorem for such functions may be stated in the form: *The Fourier development of any function having a Riemann integral is summable* (*C*1) *to the value of the function at all points except for a set of measure zero.* But Fejér's theorem does not enable us to assert anything of that sort about functions having a Lebesgue integral, since for such functions there may be no points where the limit $\frac{1}{2}[f(x + 0) + f(x - 0)]$ exists. However, Lebesgue has extended the second form of Fejér's theorem to functions integrable according to his definition [14], in a theorem which we shall now proceed to prove. We begin by establishing two lemmas.

LEMMA 3. *If* $g(t)$ *is positive or zero in the interval* $(0 \leqq t \leqq \frac{1}{2}\pi)$, *has a Lebesgue integral there, and is such that*

$$(10) \qquad \lim_{t \to 0} \frac{G(t)}{t} = \lim_{t \to 0} \frac{1}{t} \int_0^t g(t)dt = 0,$$

then we shall have

$$(11) \qquad \lim_{k \to \infty} \int_0^{\pi/2} \frac{G(t)}{t} \cdot \frac{\sin kt}{t} dt = 0.$$

Given an arbitrary positive ϵ, we choose a δ such that

$$(12) \qquad 0 \leqq \frac{G(t)}{t} < \frac{\epsilon}{4\pi} \qquad (0 < t \leqq \delta).$$

Then, δ being fixed, we choose m_1, such that

$$(13) \qquad \left| \int_\delta^{\pi/2} \frac{G(t)}{t^2} \cdot \sin kt \, dt \right| < \frac{\epsilon}{2} \qquad (k \geqq m_1),$$

which we may do in view of the Riemann-Lebesgue theorem with regard to the limiting values of the Fourier's constants of an integrable function.*

For values of $k > \pi/\delta$ we have, in view of (12),

$$(14) \qquad \left| \int_0^{\pi/k} \frac{G(t)}{t} \cdot \frac{\sin kt}{t} dt \right| < \frac{\epsilon}{4\pi} \int_0^{\pi/k} \frac{\sin kt}{t} dt$$

$$= \frac{\epsilon}{4\pi} \int_0^{\pi} \frac{\sin u}{u} du < \frac{\epsilon}{4}.$$

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* Cf. Lebesgue, Leçons sur les Séries trigonométriques, p. 61.

Furthermore, on integrating by parts,

$$(15) \quad \int_{\pi/k}^{\delta} \frac{G(t)}{t^2} \sin kt \, dt = \left[-\frac{1}{k} \cos kt \frac{G(t)}{t^2} \right]_{\pi/k}^{\delta}$$

$$+ \frac{1}{k} \int_{\pi/k}^{\delta} \frac{g(t)}{t^2} \cos kt \, dt - \frac{2}{k} \int_{\pi/k}^{\delta} \frac{G(t)}{t^3} \cos kt \, dt.$$

But

$$(16) \quad \left| \left[-\frac{1}{k} \cos kt \frac{G(t)}{t^2} \right]_{\pi/k}^{\delta} \right| \leqq \frac{1}{k} \cdot \frac{G(\delta)}{\delta^2} + \frac{1}{\pi} \cdot \frac{G(\pi/k)}{\pi/k} \, ,$$

$$(17) \quad \left| \frac{2}{k} \int_{\pi/k}^{\delta} \frac{G(t)}{t^3} \cos kt \, dt \right| < \frac{\epsilon}{2\pi k} \int_{\pi/k}^{\delta} \frac{dt}{t^2} < \frac{1}{k} \cdot \frac{\epsilon}{2\pi\delta} + \frac{\epsilon}{2\pi^2} \, ,$$

$$(18) \quad \left| \frac{1}{k} \int_{\pi/k}^{\delta} \frac{g(t)}{t^2} \cos kt \, dt \right| < \frac{1}{k} \int_{\pi/k}^{\delta} \frac{g(t)}{t^2} \, dt = \left[\frac{1}{k} \cdot \frac{G(t)}{t^2} \right]_{\pi/k}^{\delta}$$

$$+ \frac{2}{k} \int_{\pi/k}^{\delta} \frac{G(t)}{t^3} \, dt < \frac{1}{k} \cdot \frac{G(\delta)}{\delta^2} + \frac{1}{\pi} \frac{G(\pi/k)}{\pi/k} + \frac{1}{k} \cdot \frac{\epsilon}{2\pi\delta} + \frac{\epsilon}{2\pi^2} \, .$$

Combining (15), (16), (17) and (18), and taking into account (12), it is readily seen that we may choose m_2 such that

$$(19) \quad \left| \int_{\pi/k}^{\delta} \frac{G(t)}{t^2} \sin kt \, dt \right| < \frac{\epsilon}{4} \qquad (k \geqq m_2).$$

If we designate by m the greatest of the three quantities m_1, π/δ and m_2, we have from (13), (14) and (19)

$$\left| \int_0^{\pi/2} \frac{G(t)}{t} \cdot \frac{\sin kt}{t} \, dt \right| < \epsilon \qquad (k \geqq m).$$

Our lemma is therefore proved.

LEMMA 4. *If $f(x)$ is integrable (Lebesgue) in the interval $(a \leqq x \leqq b)$, $|f(x) - f(x_0)|$, where $(a \leqq x_0 \leqq b)$, is for $x = x_0$ the derivative of its indefinite integral, except perhaps at a set of points of measure zero.*

We know from a well known theorem in the theory of Lebesgue integrals that $f(x) - \alpha$, where α is any constant, will, under the conditions of our lemma, be the derivative of its indefinite integral for all points of $(a \leqq x \leqq b)$ except perhaps a set of measure zero. Let this set be represented by $E(\alpha)$, and let E_1 represent the set which is the sum of all the $E(\alpha)$ for all rational values of α. Then E_1 will also be a set of measure zero.

Let x_0 be a value of x in $a \leqq x \leqq b$ which does not belong to E_1, and let β be any irrational number. Given an arbitrary positive ϵ, we can find a rational number α so near to β that

$$(20) \qquad \Big| |f(x) - \beta| - |f(x) - \alpha| \Big| \leqq |\alpha - \beta| < \frac{\epsilon}{3},$$

whence

$$(21) \quad \left| \frac{1}{|x - x_0|} \left| \int_{x_0}^{x} |f(x) - \beta| \, dx \right| \right.$$
$$\left. - \frac{1}{|x - x_0|} \left| \int_{x_0}^{x} |f(x) - \alpha| \, dx \right| \right| < \frac{\epsilon}{3}.$$

Since, moreover, the second term of the left hand member of (21) approaches $|f(x) - \alpha|$ as x approaches x_0, we may choose a δ such that

$$(22) \quad \left| \frac{1}{|x - x_0|} \left| \int_{x_0}^{x} |f(x) - \alpha| \, dx \right| - |f(x) - \alpha| \right| < \frac{\epsilon}{3}$$
$$(|x - x_0| < \delta).$$

Combining (20), (21) and (22), we obtain

$$\left| \frac{1}{|x - x_0|} \left| \int_{x_0}^{x} |f(x) - \beta| \, dx \right| - |f(x) - \beta| \right| < \epsilon \quad (|x - x_0| < \delta).$$

Hence $|f(x) - \beta|$ is for $x = x_0$ the derivative of its indefinite integral. Since x_0 was any point not of E_1 and β was any irrational number, it follows that $|f(x) - \gamma|$, where γ is any number, is the derivative of its indefinite integral for every point of $(a \leqq x \leqq b)$ except a set of measure zero. Since for any point x_0 we may choose $\gamma = f(x_0)$, our lemma is proved.

LEBESGUE'S THEOREM. *The Fourier development of a function $f(x)$ that is periodic of period 2π and is integrable (Lebesgue) over any interval of length 2π, is summable (C1) almost everywhere to the value $f(x)$.*

Our theorem may be proved by showing that

$$(23) \qquad \lim_{n \to \infty} \left[\frac{S_n(x)}{n} - f(x) \right] = 0,$$

where $S_n(x)$ is defined by (3), is true almost everywhere. In view of (4) and Lemma 1 the expression in brackets in (23)

may be written in the form

$$(24) \quad \frac{1}{n\pi} \int_0^{\pi/2} [f(x+2t) + f(x-2t) - 2f(x)] \frac{\sin^2 nt}{\sin^2 t} \, dt$$

$$= \frac{1}{n\pi} \int_0^{\pi/2} \varphi_x(t) \frac{\sin^2 nt}{\sin^2 t} dt.$$

We will show that (24) approaches zero as n becomes infinite for all values of x for which $|\varphi_x(t)|$ is for $t = 0$ the derivative of its indefinite integral $\Phi_x(t)$. This latter will be the case for all values of x for which both $|f(x+2t) - f(x)|$ and $|f(x-2t) - f(x)|$ are for $t = 0$ the derivatives of their indefinite integrals, and in view of Lemma 4 this is true almost everywhere. Thus our theorem will be proved.

We consider then a value of x for which $\Phi_x'(0) = \varphi_x(0) = 0$, and hence we have

$$(25) \qquad\qquad \lim_{t \to 0} \left[\frac{\Phi_x(t)}{t} \right] = 0.$$

Integrating by parts in the integral on the right hand side of (24), this expression takes the form

$$(26) \quad \frac{1}{n\pi} \left[\Phi_x(t) \frac{\sin^2 nt}{\sin^2 t} \right]_0^{\pi/2} - \frac{1}{\pi} \int_0^{\pi/2} \frac{\Phi_x(t)}{\sin t} \cdot \frac{\sin 2nt}{\sin t} \, dt$$

$$+ \frac{2}{n\pi} \int_0^{\pi/2} \frac{\Phi_x(t)}{\sin t} \cdot \frac{\sin^2 nt}{\sin^2 t} dt.$$

The first term in (26) vanishes at the lower limit in view of (25) and at the upper limit takes on a value which approaches zero as n becomes infinite. From Lemma 2 and (25) the third term is readily seen to approach zero as n becomes infinite. The second term may be replaced by

$$(27) \qquad\qquad \frac{1}{\pi} \int_0^{\pi/2} \frac{\Phi_x(t)}{t} \cdot \frac{\sin 2nt}{\sin t} \, dt,$$

since the difference between the two approaches zero as n becomes infinite in view of the theorem of Riemann-Lebesgue referred to in the proof of Lemma 3. But it follows from Lemma 3 that (27) approaches zero as n becomes infinite. Hence (26), and therefore (24), has this same property for the value of x we are considering. Thus, as pointed out above, the theorem is proved.

We may also use for the summation of Fourier's series the non-integral orders of summability introduced by Knopp, Marcel Riesz and Chapman. These may be defined as follows: Let

$$(28) \quad s_n = \sum_{m=0}^{m=n} u_m, \qquad A_n^{(k)} = \frac{\Gamma(k+n+1)}{\Gamma(k+1)\Gamma(n+1)},$$

$$S_n^{(k)} = \sum_{p=0}^{p=n} A_{n-p}^{(k-1)} s_p.$$

Then if $\sigma_n^{(k)} = S_n^{(k)}/A_n^{(k)}$ approaches a limit σ as n becomes infinite, we say that the series Σu_n is summable (Ck) with sum σ. We may also define the sum of the series to be

$$(29) \qquad \lim_{\omega \to \infty} \left[\sum_{n=0}^{n<\omega} \left(1 - \frac{n}{\omega}\right)^k u_n \right]$$

whenever that limit exists. This latter definition has been shown by Riesz to be entirely equivalent to the former one.*
It was shown independently by Riesz [18] and Chapman [3] that the Fourier development of a function $f(x)$ having a Lebesgue integral is summable $(C, k > 0)$ at all points at which $\lim_{h \to 0} [\frac{1}{2}\{f(x+h) + f(x-h)\}]$ exists, to the value of that limit. It was shown by G. H. Hardy that the series is summable $(C, k > 0)$ to $f(x)$ almost everywhere [13]. Hardy's proof of his theorem is similar in method to a simplified proof of the Riesz-Chapman theorem given by W. H. Young [19] and [20]. Space is lacking to give here the details of these proofs. They depend on properties of the functions

$$C_p(t) = \frac{t^p}{\Gamma(p+1)} \left\{ 1 - \frac{t^2}{(p+1)(p+2)} \right.$$
$$\left. + \frac{t^4}{(p+1)(p+2)(p+3)(p+4)} - \cdots \right\}$$

introduced by Young. These functions are generalizations of the sine and cosine, for we have obviously $C_0(t) = \cos t$, $C_1(t) = \sin t$.

§2. Convergence Factors.

Convergence factors may be defined as a set of functions of a parameter which, when introduced as factors of the succes-

* Cf. *Comptes Rendus*, June 12, 1911.

sive terms of a series, cause a divergent series to converge,* or a series which is already convergent to converge more rapidly throughout a given range of values of the parameter. In the case of all the convergence factors used in practice; it is further true that each factor approaches unity as the parameter approaches a certain value, and that the function of the parameter defined by the series with the convergence factors approaches a limit as the parameter approaches this same value, this limit being the value of the series for convergent series and a value we find it useful to ascribe to the series in the case of a divergent series.

Thus we see that a set of convergence factors may be used to define the sum of a divergent series. This method of summation goes back to Euler, who frequently arrived at a value for a divergent series Σu_n by setting $\Sigma u_n = \lim_{x \to 1-0} \Sigma u_n x^n$. A simple illustration of this process is exhibited in the case of the series $1 - 1 + 1 - 1 + \cdots$. For this series we have

$$\lim_{x \to 1-0} \Sigma u_n x^n = \lim_{x \to 1-0} (1 - x + x^2 - x^3 + \cdots) = \lim_{x \to 1-0} \frac{1}{1 + x} = \tfrac{1}{2}.$$

This value agrees with the value of the series when summed by the mean value process.

From one point of view, practically every method of summing divergent series may be regarded as a convergence factor method. Thus in the summation by first means we seek the limit as n becomes infinite of

$$F(n) = \frac{s_0 + s_1 + \cdots + s_n}{n + 1} = u_0 + \left(1 - \frac{1}{n + 1} \right) u_1$$

$$+ \left(1 - \frac{2}{n + 1} \right) u_2 + \cdots + \left(1 - \frac{n}{n + 1} \right) u_n.$$

The expression on the right hand side may be regarded as the series obtained by introducing the convergence factors

$$f_m \left(\frac{1}{n} \right) = 1 - \frac{m}{n + 1} \qquad (m = 0, 1, 2, \cdots, n);$$

$$f_m \left(\frac{1}{n} \right) = 0 \quad (m = n + 1, n + 2, \cdots)$$

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* Some writers have employed the term convergence factor in the case of a set of factors which cause a divergent series to tend toward convergence, i. e., to become summable of a lower index.

into the series Σu_n. Here both the variables of which the convergence factors are functions take on only discrete values and we take the limit of $F(n)$ as n becomes infinite. But the principal characteristic of this type of factors lies in the fact that for any definite value of the parameter they are all zero from a certain point on. In this they differ from the type to which the name convergence factor has ordinarily been applied.

The simplest example of the latter type is to be found in the set of convergence factors mentioned above as used by Euler, namely the set $1, x, x^2, \cdots$. In connection with this set it is interesting to note that the fact we have indicated above with regard to the effect of their introduction into the series $1 - 1 + 1 - 1 + \cdots$, is not an accidental coincidence but a particular case of a general theorem due to Frobenius [8]. This theorem may be stated as follows:

FROBENIUS'S THEOREM: *If the series Σu_n is summable $(C1)$ to the value S, then the series $\Sigma u_n x^n$ will converge for $0 < x < 1$, and the function $F(x)$ which it defines for those values will be such that* $\lim_{x \to 1-0} F(x) = S$.

We shall not give the proof of this theorem, inasmuch as it is a special case of a more general theorem due to Bromwich [1]. This latter theorem includes also a number of other theorems about convergence factors due to various writers. In its most general form it applies to the introduction of convergence factors into series summable (Ck), where k is any positive integer.* We shall deal only with the simplest case where $k = 1$, but for this case we will state the theorem in somewhat more general form than it is given by Bromwich. It requires only slight changes of phraseology to fit the proof to this modified statement, and in this latter form the theorem applies directly to cases to which the other form is not applicable.

BROMWICH'S THEOREM: *If the series Σu_n is summable $(C1)$ to the value S, and the set of functions, $f_0(\alpha), f_1(\alpha), f_2(\alpha), \cdots, f_n(\alpha), \cdots$, defined for a set of values $E(\alpha)$ having at least one limit point α_0, not of the set, satisfies the conditions*

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* Bromwich's theorem has been generalized to cases where k is non-integral by Chapman and Ottolenghi. Cf. [3] and *Giornale di Matematiche di Battaglini*, vol. 49 (1911).

(A) $\displaystyle\sum_{n=p}^{n=q} n\,|\,\Delta^2 f_n(\alpha)\,|\,^* < K$ $\left(\begin{array}{l} p,\,q \text{ any integers} \\ K \text{ a positive constant} \end{array}\right)$

(B) $\displaystyle\lim_{n\to\infty} n f_n(\alpha) = 0$

(C) $\displaystyle\lim_{a\to a_0} f_n(\alpha) = 1,$

$\left.\rule{0pt}{60pt}\right\} E(\alpha),$

then the series $\Sigma u_n f_n(\alpha)$ *will converge absolutely over* $E(\alpha)$ *and the function* $F(\alpha)$ *which it defines there will be such that* $\displaystyle\lim_{a\to a_0} F(\alpha)$
$= S.\dagger$

Since we have, using the notation (28) for the case $k = 1$,

$$u_n = s_n - s_{n-1} = (S_n - S_{n-1}) - (S_{n-1} - S_{n-2})$$
$$= S_n - 2S_{n-1} + S_{n-2},$$

we obtain, if we set $S_{-2} = S_{-1} = 0$,

(30) $\displaystyle\sum_{m=0}^{m=n} u_m f_m(\alpha) = \sum_{m=0}^{m=n} (S_m - 2S_{m-1} + S_{m-2}) f_m(\alpha)$

$$= \sum_{m=0}^{m=n-1} S_m \Delta^2 f_m(\alpha) + S_n f_n(\alpha) - S_{n-1} f_{n+1}(\alpha).$$

In view of the hypothesis that Σu_n is summable $(C1)$ we may choose a positive constant C such that $|S_n| < (n+1)C$ for all values of n. Then, making use of this fact and condition (A), we see that the summation on the right hand side of (30) approaches a limit over $E(\alpha)$ as n becomes infinite. Making use of the property of S_n just employed and condition (B), it follows that the remaining two terms on the right hand side of (30) approach zero as n becomes infinite. Thus the left hand side of (30) approaches a limit also, and we have

(31) $\qquad F(\alpha) = \displaystyle\sum_{m=0}^{\infty} u_m f_m(\alpha) = \sum_{m=0}^{\infty} S_m \Delta^2 f_m(\alpha).$

Applying this identity to the series $1 + 0 + 0 + 0 + \cdots$, we obtain, since in this case $S_m = m + 1$,

(32) $\qquad f_0(\alpha) = \displaystyle\sum_{m=0}^{\infty} (m + 1)\Delta^2 f_m(\alpha).$

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* The notation $\Delta^2 f_n(\alpha)$ is used as an abbreviation for $f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)$.

† If the terms u_n are functions of a variable and Σu_n is uniformly summable throughout a certain interval, the limit of $F(\alpha)$ will be approached uniformly over that interval as α approaches α_0.

From (31) and (32) we obtain

$$(33) \quad F(\alpha) - Sf_0(\alpha) = \sum_{m=0}^{\infty} \left(\frac{S_m}{m+1} - S \right) (m+1)\Delta^2 f_m(\alpha).$$

Since Σu_n is summable $(C1)$ to S, we may choose an r such that

$$(34) \qquad\qquad \left| \frac{S_m}{m+1} - S \right| < \frac{\epsilon}{4K} \qquad\qquad (m \geqq r),$$

ϵ being an arbitrary positive quantity and K the K of condition (A). Then, r being fixed, we may in view of condition (C) choose δ such that

$$(35) \qquad \left| \sum_{m=0}^{m=r-1} \left(\frac{S_m}{m+1} - S \right) (m+1)\Delta^2 f_m(\alpha) \right| < \frac{\epsilon}{2}$$

$$(|\alpha - \alpha_0| < \delta).$$

From (33), (34), condition (A) and (35) we obtain

$$|F(\alpha) - Sf_0(\alpha)| < \epsilon \qquad (|\alpha - \alpha_0| < \delta).$$

Whence

$$\lim_{\alpha \to \alpha_0} F(\alpha) = \lim_{\alpha \to \alpha_0} Sf_0(\alpha) = S,$$

and our theorem is proved.

§3. *Applications to Problems in the Flow of Heat.*

The theorems about convergence factors are of special interest in view of the fact that they have important applications in connection with certain problems in mathematical physics. We will illustrate this by discussing a particular problem in the flow of heat.

We take the case of a finite rod of length π whose ends are maintained at zero temperature and whose surface is a nonconductor. Suppose the cross section of the bar is so small that the temperature is sensibly constant throughout it, and suppose the initial temperature is given by the function $f(x)$. We wish to determine the temperature of any point of the bar at any later moment.

Our problem reduces to the determination of a function of x and t, $v(x, t)$, such that

$$(\alpha) \quad \frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} \qquad \left(\begin{matrix} 0 < x < \pi \\ t > 0 \end{matrix} \right),$$

$$(\beta) \quad v(0, t) = 0 = v(\pi, t) \qquad\qquad (t > 0),$$

$$(\gamma) \quad \lim_{\substack{t \to +0 \\ x \to x_1}} v(x, t) = f(x_1),$$

where x_1 is any point in whose neighborhood $f(x)$ is continuous. If we assume that $v(x, t)$ may be expressed in the form $u(x)w(t)$, we find as particular solutions of (α)

$$(36) \qquad Ae^{-ka^2t} \sin ax, \quad Be^{-ka^2t} \cos ax \quad (A, B \text{ and } a \text{ constants}).$$

The second solution does not satisfy the boundary condition (β), so we reject it. The first solution satisfies (β), but will satisfy (γ) only in case $f(x) = \sin ax$. If $f(x)$ does not have this form, we naturally try to build up a sum of particular solutions of (α) of the form of the first solution in (36), $\Sigma A_n e^{-ka_n^2t} \sin a_n x$, which will approach $f(x)$ as t approaches $+ 0$. This raises the question of the possibility of expressing $f(x)$ in the form $\Sigma A_n \sin a_n x$. We know from the theory of convergent Fourier series that an arbitrary function of x, satisfying fairly wide conditions, may be expressed in a series of this form. But we also know that there exist continuous functions of x whose Fourier development diverges at points everywhere dense.

Our physical intuition tells us that there must be a solution of our problem corresponding to any original distribution of temperature that is thinkable, and therefore certainly in the case of an original distribution that is continuous. Fejér's theorem about summability $(C1)$ of the Fourier series, combined with some general theorem about convergence factors such as Bromwich's theorem, furnishes the mathematical demonstration that the series formed by introducing convergence factors of the type e^{-kn^2t} into the sine series for $f(x)$ is the desired solution.

The proof in detail is relatively brief. We have to show that the series

$$(37) \qquad \sum_{n=1}^{\infty} A_n \sin nx e^{-kn^2t} \quad \left(A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \right)$$

converges for $(t > 0; 0 \leq x \leq \pi)$, and defines in that region a function $v(x, t)$ which satisfies conditions (α), (β) and (γ). We know from the Riemann-Lebesgue theorem referred to above that A_n approaches zero as a limit as n becomes infinite, and therefore the series (37) and the various derived

series obtained by differentiating (37) term by term with re-
gard to t or x, will converge uniformly throughout the region
$(t \geq t_0 > 0, 0 \leq x \leq \pi)$. Thus we see that the series (37)
defines a function $v(x, t)$ that satisfies conditions (α) and (β).
It remains to show that this function also satisfies (γ).

It is readily seen that $v(x, t)$ will satisfy (γ) provided it ap-
proaches $f(x)$ as a limit as t approaches $+ 0$, uniformly through-
out any interval included in an interval in which $f(x)$ is con-
tinuous. Since, from Fejér's theorem, $\Sigma A_n \sin nx$ is uniformly
summable throughout any interval included in an interval of
continuity of $f(x)$, it will follow from Bromwich's theorem that
$v(x, t)$ approaches $f(x)$ uniformly throughout such an interval,
in case the convergence factors in (37) satisfy the conditions
of that theorem.

That conditions (B) and (C) are satisfied is easily seen.
Turning to condition (A), we find that $\Delta^2 e^{-kn^2 t}$ is negative
when $n \leq \sqrt{1/2kt} - 2$ and is positive when $n \geq \sqrt{1/2kt}$. Hence
this expression cannot change sign more than three times for
any value of $t > 0$, and therefore condition (A) will be satis-
fied i we can determine a positive constant such that any se-
quence of terms chosen from $\Sigma n \Delta^2 e^{-kn^2 t}$ is less in absolute value
than this constant for all values of $t > 0$.

We have

$$\sum_{n=p}^{n=q} n\Delta^2 e^{-kn^2 t} = pe^{-kp^2 t} - (p - 1)e^{-k(p+1)^2 t}$$
$$- (q + 1)e^{-k(q+1)^2 t} + qe^{-k(q+2)^2 t}$$
$$= p[e^{-kp^2 t} - e^{-k(p+1)^2 t}] - q[e^{-k(q+1)^2 t}$$
$$- e^{-k(q+2)^2 t}] + e^{-k(p+1)^2 t} - e^{-k(q+1)^2 t}$$
$$= 2kp(p + \theta_1)te^{-k(p+\theta_1)^2 t} - 2kq(q + \theta_2)te^{-k(q+\theta_2)^2 t}$$
$$+ e^{-k(p+1)^2 t} - e^{-k(q+1)^2 t} \quad \begin{pmatrix} 0 < \theta_1 < 1 \\ 1 < \theta_2 < 2 \end{pmatrix}.$$

Each of the first two terms on the right hand side is less in
absolute value than the expression $2kye^{-y}$ for some value of
$y > 0$, and since this expression and also e^{-y} remain finite for
all values of $y > 0$, it follows that condition (A) is satisfied by
the convergence factors of (37). Hence the series (37) de-
fines a function $v(x, t)$ which satisfies condition (γ), and as we
saw previously that this same function satisfies (α) and (β),
it follows that our physical problem is completely solved.

§4. *Summability of Other Developments.*

The developments in orthogonal functions of one variable that have been most extensively studied are, aside from the Fourier series, the developments in Sturm-Liouville functions and in Legendre's and Bessel's functions. The Sturm-Liouville developments offer less difficulty and yield simpler results than the other two. This is due to the fact that the differential equations which define Legendre's and Bessel's functions have singular points in the interval in which we wish to develop an arbitrary function, while the differential equation for the Sturm-Liouville functions does not. It is in the neighborhood of these singular points that the developments in terms of the former functions are more complex in their behavior and are more difficult to handle.

Very complete results with regard to the summability $(C1)$ of the Sturm-Liouville developments were obtained by Haar in his dissertation [11]. He showed in fact that the behavior of the development of any function having a Lebesgue integral was the same at any point as the behavior of the cosine development of the same function. This result follows readily from the following fundamental theorem:

HAAR's THEOREM. *If we represent by $s_n(x)$ and $\sigma_n(x)$ the sums of the first $n + 1$ terms of the Sturm-Liouville and cosine developments respectively of any function $f(x)$ having a Lebesgue integral, we have*

$$\lim_{n \to \infty} [s_n(x) - \sigma_n(x)] = 0$$

*uniformly over the interval $(0 \leqq x \leqq \pi)$.**

This theorem enables us to infer from the various results obtained with regard to the summability of the Fourier's series, corresponding results for the Sturm-Liouville developments. Thus we obtain not only the analogue of Fejér's theorem, as pointed out by Haar, but also the analogues of Lebesgue's theorem, the Riesz-Chapman theorem, and Hardy's theorem.

The first study of the summability of the developments in Legendre's functions was made in 1908 by Fejér who established summability $(H2)$, or what is equivalent $(C2)$, at every

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* If the interval of definition of the Sturm-Liouville functions is taken as some interval (a, b) differing from the interval $(0, \pi)$, we readily reduce that case to the above by a change of variable in the differential equation and boundary conditions by means of which the functions are defined.

point in the interval $(-1 \leqq x \leqq 1)$ at which the function is continuous or has a finite jump, provided the function is absolutely integrable [6]. In 1911 Haar proved the summability $(C1)$ of the development at interior points of the interval $(-1 \leqq x \leqq 1)$ for the case where the function developed is continuous throughout the whole interval [12]. During the same year Chapman independently established summability $(C, k > 1)$ for all points of the interval at which the function is continuous or has a finite jump, provided the function has a Lebesgue integral over the interval. He further obtained summability $(C, k > \frac{1}{2})$ at the end points and $(C, k > 0)$ at interior points for the case where the function satisfies certain additional restrictions as to the possession of limited variation [4]. In 1913 Gronwall obtained summability of the same orders with less restriction on the function developed. His requirement for the case of interior points is that $f(x)$ and $(1 - x^2)^{(k/2)-(1/4)}f(x)$ should be absolutely integrable in $(-1, 1)$. For the end points he requires the absolute integrability of $f(x)$ and a further condition on the function in the neighbor-..ood of the end point opposite to the one for which summability is established. This condition, in the case of summability at $+1$, demands that

$$\int_{-1}^{-1+\delta} f(x')\sigma_n\big((2n+1)P_n(x)P_n(x')\big)dx',$$

where $P_n(x)$ represents the Legendrian of the nth order and $\sigma_n^{(k)}(u_n) = S_n^{(k)}/A_n^{(k)}$, $S_n^{(k)}$ and $A_n^{(k)}$ being defined by equation (28), should, for some value of $\delta > 0$, approach zero as n becomes infinite [10].

In the case of the developments in Bessel's functions, summability $(C1)$ at points in the interval $(0 < x < 1)$ where the function developed is continuous or has a finite jump, was established by the writer in 1908 [15] for the case of a function that is finite and integrable. Summability $(C1)$ for points of the same nature in $(0 < x \leqq 1)$, was established by W. B. Ford in his recent book on divergent series and summability [7] for the case of a function $f(x)$ that becomes infinite at a finite number of points while $\sqrt{x}f(x)$ remains absolutely integrable. Summability $(C, k > 0)$ at points in the interval $(0 < x \leqq 1)$ at which the function is continuous or has a finite jump for a function $f(x)$ such that $\sqrt{x}f(x)$ has a Lebesgue

integral, and summability $(C\tfrac{1}{2})$ for $x = 0$, provided $f(x)$ has a Lebesgue integral, is continuous at the origin and is such that $\left| \left(f(x) - f(0) \right)/x^{\gamma} \right|$ is less than a positive constant for some value of $\gamma > \tfrac{1}{2}$ and values of x in the neighborhood of zero, have been established by the writer in two papers that have been presented to the Society but not as yet published.* Thus we see that here as in the case of the developments in Legendre's functions, we have to put more restriction on the function developed at the point at which the differential equation corresponding to the functions in terms of which we develop has a singular point; moreover, even then we do not get summability of as low an order. The analogy between the two cases suggests the existence of a general theory† with regard to the summability of developments in orthogonal functions that satisfy linear differential equations with singular points, which will include the important features of both cases. As far as the writer is aware this general theory is yet to be developed.

§5. *The Summability of Double Series.*

All the different methods of summation used in connection with simple series can be readily generalized so as to furnish methods for summing double series. We shall discuss here only the extension of the simplest method, i. e., summation by arithmetic means of the sums, to double series. We consider the double series Σa_{ij}, and we set

$$s_{mn} = \sum_{i=0,j=0}^{i=m,j=n} a_{ij}, \quad S_{mn} = \sum_{i=0,j=0}^{i=m,j=n} s_{ij},$$

$$\sigma_{mn} = S_{mn}/(m + 1)(n + 1).$$

Then if $\lim_{m,\,n\to\infty} \sigma_{mn}$ exists and is equal to σ, we say that the series Σa_{ij} is summable $(C1)$ to the value σ. It is easy to establish the consistency of this definition with the Pringsheim definition of convergence, for series such that $|s_{mn}| < C$, a positive constant, for all values of m and n. Moreover, in that case it will also be true that $|\sigma_{mn}| < C$ for all values of m and n.

The theorem of Frobenius for simple series has been extended to double series by Bromwich and Hardy [2], and Brom-

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* For abstracts of these papers, see BULLETIN, vol. 24, pp. 65 and 422.

† Cf. a principle of generalization laid down by E. H. Moore, Introduction to a Form of General Analysis, p. 1.

wich's theorem for simple series has been extended to double series by the writer [16].   Fejér's theorem about the summability of the ordinary Fourier series has been extended to the double Fourier series, for points of continuity* of the function developed, by W. H. Young [21] and the writer [16], and for points of discontinuity of certain types by the writer [17].   These extensions, taken in connection with the extension of the theorem on convergence factors, enable us to discuss certain problems in mathematical physics in which double Fourier series occur, in a manner analogous to the discussion of the problem in the flow of heat considered earlier in the paper.   Such a discussion may be found in [16].

The consideration of the summability of the double Fourier series naturally suggests the consideration of the summability of double series involving other orthogonal functions. Furthermore, the extension of the conception of summability to double series leads naturally to its further extension to triple series and to multiple series of any order.   In particular we might study the summability of the triple Fourier series and thus obtain results having important applications to mathematical physics, analogous to those mentioned in connection with the ordinary and double Fourier series.   Many other special studies readily suggest themselves.   Thus we see that the work already done on the summability of developments in orthogonal functions is only a beginning, and that a large unexplored field remains to be investigated.

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MODULAR SYSTEMS.

The Algebraic Theory of Modular Systems. By F. S. MACAULAY. [Cambridge Tracts in Mathematics and Mathematical Physics, No. 19.] Cambridge University Press, 1916. xiv + 112 pp.

A MODULAR system is an infinite aggregate of polynomials in n variables $x_1, x_2, \cdots, x_n$, defined by the property that if F, F_1, F_2 belong to the system, $F_1 + F_2$ and AF also belong to the system, where A is any polynomial in $x_1, x_2, \cdots, x_n$. Hence if $F_1, F_2, \cdots, F_k$ belong to a modular system so also does $A_1F_1 + A_2F_2 + \cdots + A_kF_k$, where $A_1, A_2, \cdots, A_k$ are arbitrary polynomials in $x_1, x_2, \cdots, x_n$. In the algebraic theory (to which this tract is devoted) polynomials such as F and aF, where a is a quantity not involving the variables, are regarded as the same polynomial.