

\mathfrak{B}^m . The system of neighborhoods \mathfrak{B}^m for fixed m covers Ω and may be replaced by a finite subsystem,

$$\mathfrak{B}^{m_1}, \mathfrak{B}^{m_2}, \dots, \mathfrak{B}^{m_k};$$

such that each point of Ω is interior to some class \mathfrak{B}^{m_k} and each class \mathfrak{B}^{m_k} is a neighborhood of a point Q^{m_k} of the class Ω . Let \mathfrak{C} be the class of all points Q^{m_k} . Since every point P of Ω is interior to some set \mathfrak{B}^{m_k} , it follows from condition (4) that Q^{m_k} is contained in the neighborhood \mathfrak{B}^m of P . Therefore P is a limiting point of the class \mathfrak{C} .

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This article was in type before the writer learned of the existence of an article by Fréchet (*Bulletin de la Société mathématique de France*, volume 35, 1917), in which it is shown that the *closure of derived classes* is a consequence of the Heine-Borel property in the case of a general system \mathfrak{L} . Theorem 3 of the present paper is a generalization of this result.

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INTEGRALS AROUND GENERAL BOUNDARIES.

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THE concept of a boundary integral has been extended to curves of the type $x = x(t)$, $y = y(t)$, where $x(t)$, $y(t)$ are absolutely continuous functions of a parameter t . In this case the curves are more or less simple and have tangents "nearly everywhere." In applications to physics however the boundary must be considered rather as a boundary of a set (in the sense of the theory of point sets). The boundary will be, in general, a collection of points without definite tangents at all. This paper sets out a method by which such boundary integrals can be defined under certain restrictions placed on the two integrand functions u , v . The method depends on the concept of absolutely additive functions of sets.* The writer believes that these restrictions could be lightened and that there is a wide field here for further investigation.

* J. Radon, *Wiener Sitzungsberichte*, vol. 122 (1913), p. 1295. W. H. Young, *Proceedings London Math. Society*, vol. 13 (1914), p. 109.

Statement of Problem.—Given any set E measurable Borel, and its boundary $B(E)$, contained in a closed fundamental interval J ($0 \leq x \leq 1, 0 \leq y \leq 1$); given also two functions $u(x, y), v(x, y)$ summable in the interval J ; to define

$$\int_{B(E)} u dx + v dy.$$

Note.—The boundary integral is taken in such a sense that on a rectangle for the side with the lesser value of y the integral is taken in the direction of x increasing. As the axes are usually drawn this corresponds to a counter-clockwise sense.

Definitions and Restrictions.—

- R 1. Let the total variation of $u(x, y)$, varying y , be $\lambda(x)$, where $\lambda(x)$ is finite nearly everywhere in x and summable in ($0 \leq x \leq 1$).
- R 2. Let the total variation of $u(x, y)$, varying x , be $\mu(y)$, where $\mu(y)$ is finite nearly everywhere in y and summable in ($0 \leq y \leq 1$).

We shall consider in the first place rectangles r with sides parallel to the axes. Then

$$\int_{B(r)} u dx + v dy = \int_r d\alpha(x, y)$$

can be proved to be an absolutely additive function of rectangles r , and we may define

$$\int_{B(E)} u dx + v dy = \int_E d\alpha(x, y).$$

If $d\alpha(x, y)$ is an absolutely additive function of rectangles we can by Radon's method define $\int_E d\alpha(x, y)$ uniquely for any set E measurable Borel contained in J . All that is needed then is to prove that $\int_{B(r)} u dx + v dy$ is an absolutely additive function of rectangles r and to define

$$\alpha(x, y) = \int_{B(r')} u d\xi + v d\eta,$$

where r' is the rectangle ($0 < \xi < x, 0 < \eta < y$).

Proof. By R_1 , the total variation of $u(x, y)$ in ($0 \leq y \leq 1$)

is $\lambda(x)$. Denote the total variation in $(0 \leq \eta \leq y)$ by $\lambda(x)\theta(x, y)$ when $\lambda(x)$ is finite. $\lambda(x)$ is non-negative, $\theta(x, y)$ is non-negative and a non-decreasing function of y taking the value 0 when $y = 0$, and 1 when $y = 1$. In particular it is a limited measurable function of x . Define

$$\begin{aligned}\sigma_1(x, y) &= \int_0^x \lambda(x)\theta(x, y)dx. \\ \int_r d\sigma_1(x, y) &= \sigma_1(x_1, y_1) + \sigma_1(x_2, y_2) - \sigma_1(x_1, y_2) - \sigma_1(x_2, y_1) \\ &= \int_{x_1}^{x_2} \lambda(x)dx[\theta(x, y_2) - \theta(x, y_1)]\end{aligned}$$

is a finite non-negative additive function of rectangles. Then for any set E measurable Borel

$$\int_E d\sigma_1(x, y) \text{ is defined and } \leq \int_J d\sigma_1(x, y) \text{ or } S_1.$$

By R_1 ,

$$|u(x, y_1) - u(x, y_2)| \leq \lambda(x)[\theta(x, y_2) - \theta(x, y_1)].$$

$u(x, y)$ is summable in (x, y) or is summable in x for nearly all values of y . Let y_0 be one of the values for which it is summable. Then

$$|u(x, y)| \leq |u(x, y_0)| + \lambda(x)|[\theta(x, y) - \theta(x, y_0)]|$$

or $u(x, y)$ is summable in x for *all* values of y .

$$\begin{aligned}\left| \int_{B(r)} u dx \right| &= \left| \int_{x_1}^{x_2} [u(x, y_1) - u(x, y_2)] dx \right| \\ &\leq \int_{x_1}^{x_2} \lambda(x)[\theta(x, y_2) - \theta(x, y_1)]\end{aligned}$$

or

$$\int_r d\sigma_1(x, y).$$

Hence for any set $\sum_{i=1}^n r_i$ of non-overlapping rectangles

$$\begin{aligned}\sum_{i=1}^n \left| \int_{B(r_i)} u dx \right| &\leq \int_{\sum r_i} d\sigma_1(x, y) \\ &\leq S_1.\end{aligned}$$

Therefore

$$\sum_{i=1}^{\infty} \int_{B(r_i)} u dx$$

is absolutely convergent.

Moreover $\int_{B(r)} u dx$ is an additive function of rectangles r , for if two or more rectangles have some parts of their boundaries in common (but do not overlap), the integrals along these parts being taken in opposite directions will annul each other.

If we define

$$\alpha_1(x, y) = \int_{B(r')} u dx, \quad r' = (0 \text{ to } x, 0 \text{ to } y),$$

$$\int_{B(r)} u dx = \int_r d\alpha_1(x, y)$$

defines an absolutely additive function of rectangles. Similarly for

$$\int_{B(r)} v dy = \int_r d\alpha_2(x, y)$$

and therefore also for

$$\int_{B(r)} u dx + v dy = \int_r d\alpha(x, y),$$

where $\alpha(x, y) = \alpha_1(x, y) + \alpha_2(x, y)$.

This was to be proved, and it follows that we can define

$$\int_{B(E)} u dx + v dy = \int_E d\alpha(x, y).$$

More generally, the same method could be used if it can be proved that $\int_{B(r)} u dx + v dy$ is an absolutely additive function of rectangles; the difficulty is to state the required conditions as conditions on u and v directly. That is the reason for the introduction of R_1, R_2 , which are sufficient but probably not necessary.

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