ADDITIVE FUNCTIONS OF A POINT SET.

Intégrales de Lebesgue, Fonctions d'Ensemble, Classes de Baire. Par C. DE LA VALLÉE POUSSIN. Paris, Gauthier-Villars, 1916. viii + 154 pp.

THE general notion of an additive function of a point set is one of the most important introduced by the present French school of mathematicians. It is due to H. Lebesgue. An additive function of a point set is one whose value on a sum of sets is the sum of its values on each term. These terms, the number of which may be infinite, have in pairs no common element. The volume under review (one of the Borel monographs) is devoted to the theory of additive functions of a point set; it contains the matter of lectures delivered at the College of France between December, 1915, and March, 1916. The author had already treated the problem in his Harvard lectures, which were partially published in 1915 in the *Transactions of the American Mathematical Society.* Some of the results had also appeared in volume II of his Cours d'Analyse, third edition. In the present volume the treatment is rendered distinctly more complete and satisfying through the use of new methods and the derivation of new results.

In these lectures we have a careful analysis of the notion of additivity and the derivation of the consequences, singularly precise and interesting, which follow from the sole property of a function implied in this notion.

The simplest and earliest known additive function of a point set is its measure. The definition was given by Borel in 1898. It furnishes the point of departure for the entire theory of additive functions. It is this function which is considered in the beginning of the first of the three (nearly equal) parts of the present monograph. (Lebesgue integrals are treated in the latter portion of this part.) The measure is a non-negative function whose value is given on certain point sets called elementary figures and is then to be defined on other sets in such way as to satisfy the requirement of being additive. The point sets on which this requirement is satisfied are then the measurable sets.

If a function possesses the property of being additive only for a finite number of terms it is said to be additive in a restricted sense; if for an enumerable infinitude of terms it is said to be additive in a complete sense. The essential progress obtained by Borel and Lebesgue in the theory of measure consists in their having realized additivity in the complete sense. The first idea of this theory is due to Borel. The proper work of Lebesgue commences only with definite integrals.

The question of measure from the point of view of Borel's contribution gives rise to the following difficult problem: "a function being given on certain particular point sets, such as the elementary domains, does there exist an additive function which coincides with the preceding on these domains?" It is this fundamental question which is resolved in the second part of these lectures. One reaches the conclusion that there is complete equivalence, under the condition of continuity, between an additive function of a point set and a function $f(x)$ of bounded variation.

A function $g(e)$ of a point set e is said to be continuous if its value approaches zero with the diameter of e, and to be absolutely continuous if its value approaches zero with the measure of e. Among the additive functions of a point set the most important are those which are absolutely continuous. The indefinite integrals of Lebesgue possess these two properties of additivity and absolute continuity and are characterized by them. The derivatives of absolutely continuous functions of a point set are studied (also in the second part) by the new method of networks (réseaux) already utilized in the Harvard lectures. We return later to a discussion of this method.

The questions mentioned in the foregoing paragraphs are throughout of a metric nature. Questions of a more exclusively descriptive kind, but intimately related to the foregoing, are treated in the third part. Here the author takes up the separation of functions into the successive classes of Baire, simplifying and completing the exposition of this interesting theory by the introduction of new methods and the addition of new results. The relation of Baire's classification to the other matters treated in the book is brought out by showing the identity of the class of functions measurable in the sense of Borel and the class of functions contained in Baire's classes of all orders.

Thus the questions treated in this work belong to the recent theory of functions of which the founders are Borel, Baire and Lebesgue. In the course of his profound researches Baire has set off a real functional domain which appears to suffice for all the needs of analysis and seems to be such that every generalization is condemned beforehand to be vain and sterile. The functions of this domain possess precise common properties. The general methods of analysis are applicable to them; and their theory, already rich in results, may be considered as the *general theory of functions of a real variable.*

Thus we have here a fundamental advance from the general point of view of the philosophy of mathematics. In this progress the work of Lebesgue has been central. More than any other, Lebesgue has contributed to put in evidence the unity of this theory of functions of real variables and to beautify it by bringing out that aesthetic character which delights the artistic spirit of the pure mathematician. The author has achieved his aim in the book if the reader recognizes this character in his pages.

Such are the general features of this monograph in which there is much to please one and nothing to annoy. It remains to give a brief account of the novelties in the author's methods and results.

In the first part, which deals with measurable sets and the integrals of Lebesgue, the extent of novelty is not so great as in the other two parts. Here the author treats general notions about point sets, measure of sets and measurable functions, and Lebesgue integrals, devoting a chapter to each of the three topics. On page 10 is given a generalization of the usual definitions of interior, exterior, and frontier points of a set (see also page 105). But the chief novelty to which attention may be called is in the notion of the characteristic function of a point set, a notion already utilized by de la Vallée Poussin in his *Transactions* memoir.

Let E be a point set in a space $x, y, \cdots$ of one or more dimensions and let CE denote the set complementary to E in such a space. Then the characteristic function or the characteristic of the point set E is the function $\varphi(x, y, \cdots)$ which has the value unity at each point of E and the value zero at each point of CE. It is clear that $1 - \varphi$ is the characteristic of CE. If $\varphi_1, \varphi_2, \cdots$ are the characteristics in order of E_1, $E_2, \cdots$, these sets having in pairs no common elements, then

$\varphi_1 + \varphi_2 + \cdots$ is the characteristic of $E_1 + E_2 + \cdots$. Also, $\varphi_1\varphi_2\cdots$ is the characteristic of $E_1E_2\cdots$ without condition on $E_1, E_2, \cdots$. To find the characteristic of the sum $E_1 + E_2 + \cdots$, when no condition is put on the component sets, observe that this sum is the complement of the set $CE_1 \cdot CE_2 \cdot CE_3 \cdots$; hence its characteristic is

$$1 - (1 - \varphi_1)(1 - \varphi_2)(1 - \varphi_3)\cdots.$$

Through the use of characteristic functions the problem of infinite sums and products of point sets is readily reduced to that of the passage to a limit so that certain fundamental properties of such sums and products may at once be read off by means of corresponding properties of limits.

It is not difficult to show (see page 36) that the characteristic function of a point set which is measurable in the sense of Borel belongs to one of Baire's classes. This leads to a natural classification of point sets which are measurable in the sense of Borel, the class number of the set being the same as that of the corresponding characteristic function in the classification of Baire. Certain relations among the classes of point sets then follow at once from corresponding relations among the classes of Baire.

In approaching the theory of measure in the second chapter, de la Vallée Poussin develops first of all the theory of measure for enumerable sums of intervals and closed sets of points and then through the aid of this theory defines the measures of more general sets. In this connection it is of interest to note that Bliss (this BULLETIN, volume 24 (1917), pages 1–47) has returned to the more direct methods of Borel and Lebesgue which found the theory of measure entirely upon the measurability of enumerable sets of intervals. Through aid of the improved methods of de la Vallée Poussin, Bliss is able to approach the subject in a way which is especially concise and clear. At the same time he establishes without additional complication the foundations of the theory of the positive additive functions of a point set, of which the measure function is a special case. The reader of de la Vallée Poussin will find it useful to compare the improved exposition of Bliss.

The second part of the monograph deals in general with additive functions of a point set, its three chapters having in order the following titles: general notions concerning derivatives and networks; absolutely continuous and additive functions of a point set, indefinite integrals; additive functions of a normal set. This part is rich in novelty. One meets here the term symmetric derivatives (page 59), the particularly fruitful notion of networks (pages 61 ff.) and of derivatives on a network, and again (page 74) the major and minor functions of a summable function $f(x)$ which de la Vallée Poussin first introduced in 1909 in the second edition of volume I of his Cours d'Analyse. The greater part of Chapter VI is based on the author's *Transactions* memoir; the exposition is simplified and the results are frequently generalized or rendered more precise.

The new method of networks already employed by de la Vallée Poussin in his Harvard lectures is now rendered much more useful through the notion of conjugate networks. By aid of this, certain demonstrations are made in a simpler and more natural manner than before and the form of the exposition is improved in many respects. The novelty associated with this method constitutes one of the two central features of the monograph and is perhaps the most interesting illustration in it of the author's insight into the problems with which he is dealing. Fortunately a clear, excellent and readable connected exposition of these new ideas is given on pages 61–66, so that it is unnecessary for us to attempt a summary here in our limited space. The reader who wishes to follow up some of the main uses of the method elsewhere in the monograph may consult pages 67–82 and 96–104.

The second central feature of the monograph is to be found in the third part in the treatment of Baire's classification of functions. In his thesis (1899) Baire demonstrated the following theorem: A necessary and sufficient condition that a discontinuous function f shall be of class 1 on a bounded and perfect set P is that it shall be pointwise discontinuous on every perfect set Q contained in P. This is called Baire's theorem. It is intimately associated with the problem of Baire, namely: Given a function f of class 1, to construct a sequence of continuous functions $f_1, f_2, \cdots$ such that f_n approaches f as n approaches infinity.

Baire proved the last preceding theorem by the direct process of resolving the corresponding problem. The first demonstration was confined to a function of a single variable. Lebesgue (1899) proved the theorem for a function of several variables by a process of reduction to the case of a single variable. Baire extended his original method so as to render it applicable to functions of several variables. Finally he gave a systematic exposition of his theory in 1905 in his Leçons sur les Fonctions discontinues.

The methods of Baire make appeal to the transfinite. Lebesgue* has given demonstrations of Baire's theorem not requiring the intervention of the transfinite. He states the condition in the form: A necessary and sufficient condition that a discontinuous function f shall be of class 1 on a bounded and perfect set P is that the set of points where $f > A$ and the set where $f < A$ shall be sums of closed sets for every A. The methods of Lebesgue do not afford a solution of Baire's problem.

In the present monograph we have a new demonstration of Baire's theorem and a new and simple resolution of Baire's problem. The method of exposition is a sort of synthesis of those employed by Lebesgue and Baire. It completes those of Lebesgue and, by the introduction of an auxiliary theorem, notably simplifies that of Baire.

In the theorems of Baire and Lebesgue we have two very different expressions of the condition that a function shall be of class 1. If one seeks to generalize these theorems and to obtain analogous distinctive characteristics of functions of general class α ($\alpha > 1$), he finds that Baire's form generalizes only imperfectly (see page 144) while an adequate and complete generalization of Lebesgue's form is available (see page 141). Lebesgue's method of treatment is here followed in the main, with improvements in certain parts and the introduction of some novel results.

For the statement of the generalization one needs to employ the notion of sets O and F of Lebesgue with respect to a bounded and perfect set P. De la Vallée Poussin puts these definitions in the following form: *A set E is an F of class α, if there exists a function θ, defined on P and of class $\leqq \alpha$, such that the set E coincides with the set of points of P on which $\theta = 0$; a set E is an O of class α, if there exists a function θ, defined on P and of class $\leqq \alpha$, such that the set E coincides with the set of points of P on which $\theta \neq 0$.*

The generalized condition of Lebesgue may now be stated in the following form: A necessary and sufficient condition

* Borel's Leçons sur les Fonctions de Variables réelles et leur Représentation par les Séries de Polynomes, Note II; *Bulletin de la Société mathématique de France*, 1905.

that a function f (finite or not) shall be of class $\alpha > 0$ on P is that for every constant a the set where $f > a$ and that where $f < a$ shall be sets O of class α and shall not be both of class less than α; or, what amounts to the same thing, that the (complementary) set where $f \leqq a$ and that where $f \geqq a$ shall be sets F of class α and not both of class less than α.

The monograph closes with a demonstration (at bottom agreeing with that of Lebesgue but expressed in somewhat more elementary form) of the theorem which asserts the existence of functions of all of Baire's classes.

In view of the numerous definitions of integration which have been given and the present state of unrest in this range of ideas, one desires to consider Lebesgue integrals not merely in themselves but also in their relation to other types of integrals. Such a comparative treatment does not fall within the scope of de la Vallée Poussin's monograph. Fortunately a careful analysis of this sort is now (readily) accessible in Hildebrandt's recent paper (this BULLETIN, volume 24 (1917–1918), pages 113–144, 177–202).

In the monograph under review and in the two articles by Bliss and by Hildebrandt already referred to we have recent treatments of the general problem of integration which supplement and complete each other in a useful way. De la Vallée Poussin develops the theory of Lebesgue integrals in intimate association with several fundamental matters to which at bottom they are closely related, so that through this monograph one may obtain a view of this domain of mathematics in its proper relation to other fields and in appropriate perspective as to its own parts among themselves. Bliss gives a particularly compact, but at the same time clear, development of the theory of Lebesgue integrals and the intimately associated theory of measure, the march of ideas in his treatment being singularly direct and leading the reader to the goal without loss of energy due to indirect processes and hence with a minimum of effort. Hildebrandt's purpose is to discuss briefly the several definitions of integration and particularly to consider their relations one to another. He gives careful attention to the distinctive features of each definition, to the question of equivalence, and to the problem of further generalization.

Equivalences among integrals are of two types. The first may be called complete equivalence; it is one in which the two symbols of integration operate upon the same class of functions and give the same integral value for the same function. The other is in reality a pseudo-equivalence expressed in the fact that an integral of one kind may be transformed into an integral of another kind, the functions integrated in the two cases being different. A careful and instructive analysis of these equivalences is given by Hildebrandt.

To one result of this analysis it is desirable to have attention sharply directed. The Stieltjes integral seems destined to play in the future a rôle of central importance in processes of integration and summation. The Lebesgue integral when introduced received almost immediate attention and recognition and found its way rapidly into the main current of mathematical thought; but the Stieltjes integral has been singularly neglected notwithstanding its inherent simplicity and naturalness. Through the summary of its properties given by Hildebrandt and the applications mentioned one is convinced of its central importance and is led to expect it to assume a new place in mathematical thought. In this connection it is of particular interest to note also Hildebrandt's extension of the Stieltjes integral modelled on the Lebesgue extension of the Riemann integral.

We conclude with the following list of misprints in de la Vallée Poussin's monograph: page 20, last line, write $m(F_1 + F_2) = mF_1 + mF_2$; page 28, second theorem, write $(f \geqq A)$ instead of $(f \geqq a)$; page 34, line 16, write $\omega 2$ instead of ω_2; page 96, end of second paragraph, write Df instead of DF; page 126, line 9 below, write "Une" instead of "Uue"; page 133, line 2, write "de classe $\leqq \alpha$."

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SHORTER NOTICES.

Leçons de Mathématiques Générales. Par L. ZORETTI. Paris, Gauthier-Villars, 1914. 8vo. xvi + 753 pp.

Exercices numériques de Mathématiques. Par L. ZORETTI. Paris, Gauthier-Villars, 1914. 8vo. xv + 125 pp.

THE author of this text and its accompanying set of exercises is well qualified for the task. Formerly instructor in the