

THE OCTOBER MEETING OF THE AMERICAN
MATHEMATICAL SOCIETY.

THE one hundred and eighty-sixth regular meeting of the Society was held in New York City on Saturday, October 28, 1916. The attendance at the morning and afternoon sessions included the following forty-two members:

Professor M. J. Babb, Dr. F. W. Beal, Mr. D. R. Belcher, Professor E. W. Brown, Dr. Emily Coddington, Professor F. N. Cole, Professor Elizabeth B. Cowley, Professor Louise D. Cummings, Dr. H. B. Curtis, Professor L. P. Eisenhart, Mr. G. W. Evans, Dr. C. A. Fischer, Professor T. S. Fiske, Dr. C. H. Forsyth, Professor O. E. Glenn, Dr. T. H. Gronwall, Professor C. C. Grove, Professor M. W. Haskell, Professor H. E. Hawkes, Professor Dunham Jackson, Dr. I. L. Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Dr. J. R. Kline, Mr. E. H. Koch, Jr., Mr. Harry Langman, Professor H. B. Mitchell, Professor R. L. Moore, Mr. G. W. Mullins, Professor H. W. Reddick, Mr. J. F. Ritt, Mrs. J. R. Roe, Dr. Caroline Seely, Professor L. P. Sicheloff, Professor D. E. Smith, Professor P. F. Smith, Professor Oswald Veblen, Dr. J. H. Weaver, Dr. Mary E. Wells, Professor H. S. White, Mr. J. K. Whittemore, Professor J. W. Young.

The President of the Society, Professor E. W. Brown, occupied the chair. The Council announced the election of the following persons to membership in the Society: Mr. A. C. Bose, Calcutta, India; Professor L. C. Emmons, Michigan Agricultural College; Professor A. M. Harding, University of Arkansas; Dr. W. L. Hart, Harvard University; Dr. J. R. Musselman, University of Illinois; Mr. S. Z. Rothschild, Sun Life Insurance Company, Baltimore, Md.; Professor Pauline Sperry, Smith College. Six applications for membership in the Society were received.

The Council submitted a list of nominations for officers and other members of the Council to be elected at the annual meeting. Committees were appointed to arrange for a joint session at the annual meeting with other scientific bodies and for the summer meeting of 1917.

The following papers were read at the October meeting:

(1) Mrs. J. R. ROE: "Interfunctional expressibility problems of symmetric functions."

- (2) Professor E. D. ROE, JR.: "A geometric representation."
 (3) Professor E. D. ROE, JR.: "Studies of the Kreisteilungsgleichung and related equations."
 (4) Professor E. D. ROE, JR.: "The irreducible factors of $x^{n-1} + x^{n-2} + \dots + 1$."
 (5) Professor H. B. MITCHELL: "On the imaginary roots of a polynomial and the real roots of its derivative."
 (6) Dr. J. H. WEAVER: "Some properties of parabolas generated by straight lines and circles."
 (7) Professor F. N. COLE: "Complete census of the triad systems in fifteen letters."
 (8) Professor O. E. GLENN: "Translation surfaces associated with line congruences."
 (9) Professor O. E. GLENN: "Methods in the invariant theory of special groups, based on finite expansions of forms."
 (10) Professor R. L. MOORE: "A theorem concerning continuous curves."
 (11) Dr. J. R. KLINE: "The converse of the theorem concerning the division of a plane by an open curve."
 (12) Mr. H. S. VANDIVER: "Note on the distribution of quadratic and higher power residues."
 (13) Mr. H. S. VANDIVER: "The generalized Lagrange indeterminate congruence for a composite ideal modulus."

Professor Roe's first two papers were read by Mrs. Roe; his third paper, Professor Glenn's first paper, and Mr. Vandiver's papers were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. If α is a complex of numbers, b the elementary symmetric functions of the α 's, S the power sums, and t the complete symmetric functions, then, of the complex α , Mrs. Roe undertakes to investigate the relations of the four kinds of functions, viz.: the monomial symmetric functions, the power sums, the elementary symmetric functions and the complete symmetric functions. If the four kinds of functions are taken two at a time, six combinations arise, and a consideration of the expressibility of each of a pair in terms of the other leads to a systematic study of twelve tables of coefficients.

The principal object of the investigation is to derive the properties of, and interfunctional relations and general formulas for, the coefficients in these tables.

The problems of the t 's and the α 's and also those of the S 's and the α 's are especially considered in this paper. Tables up to weight ten illustrate the results obtained.

2. Professor Roe adjoins the complex plane to the x axis of the real plane for the graphical realization of the complex values, when such exist, of y in $y = f(x)$, and also for the purpose of representing some real curves in space by a single equation in x and y . The complex plane is adjoined so that its origin is always in the axis of x with its axis of positive reals parallel to the positive direction of the axis of y , in fact in the real plane, as the complex plane slides along always perpendicular to the x axis at the distance x as x changes. By this representation every curve has an actual locus from $-\infty$ to $+\infty$. The method is applied to the function

$$y = \{x^{x-1}(x-1)^x\}^{1/(1-2x)}.$$

In the usual simplest real representation this consists of a curve in the real plane between $x = -\infty$ and $x = 0$, and $x = 1$ to $x = +\infty$, but between $x = 0$ and $x = 1$ consists only of discrete points. But if the complex value of y corresponding to a value of x is laid off on the complex plane adjoined to the x axis at the distance x , the curve between 0 and 1 becomes multiple valued for each simplest branch, continuous, and consists of an infinite number of spirals which pierce the real plane in the discrete points and all lie on a real surface of revolution $y^2 + z^2 = [\text{mod } \{x^{x-1}(x-1)^x\}^{1/(1-2x)}]^2$ where x, y, z are the coordinates of real space of three dimensions. Finally the general equation in x and y alone of all real spirals in space of three dimensions winding around the axis of x is shown to be

$$y = f_1(x)e^{f_2(x)i}.$$

3. In this paper Professor Roe studies the four equations $\cos n\varphi \pm \cos (n+1)\varphi = 0$, $\cos (n-1)\varphi \pm \cos (n+1)\varphi = 0$, obtaining the number of and expressions for their distinct roots and applying the results to some geometrical constructions. The "Kreisteilungsgleichung" in algebraic form (derivable from the second equation) is studied and an explicit expression is obtained for the general coefficient of any term in it. The same is also done for a like algebraic equation derivable from the first equation above.

4. In this paper Professor Roe considers the irreducible factors of $f(x, n) \equiv x^{n-1} + x^{n-2} + \dots + 1$ by means of sums of powers of the roots of $(x - 1)f(x, n) = 0$.

5. The object of Professor Mitchell's paper is the deduction of simple criteria which may enable us to infer, without laborious calculation, the approximate position of a pair of imaginary roots of a polynomial with real coefficients, when the existence of imaginary roots is revealed through the vanishing of the derivative, (a) beyond the interval between the greatest and the least real root of the polynomial, (b) more than once in an interval between two consecutive real roots. Under the first assumption, upper and lower limits are determined which depend only upon the degree of the polynomial and the distance between a real root of the derivative and the adjacent real root of the polynomial. Under the second assumption it is shown that if the polynomial have not more than 8 imaginary roots, the real part of at least one pair must lie in the interval where the derivative vanishes more than once, and that the coefficient of i must be less in absolute value than the extent of that interval. When the polynomial has more than 8 imaginary roots, but when the number of real roots on one side of the interval is not greater than 8, it is shown that not all of the imaginary roots can have their real parts without and on one side of the interval—this restriction thus applying to all polynomials with real coefficients whose degree is less than 20.

The criteria are particularly suited to graphical analysis, as they involve only the approximate positions of the real roots and bend points.

6. Starting with the well-known fact that the locus of the center of a circle tangent to a given straight line and a given circle is two parabolas (one of which may be degenerate), Dr. Weaver proves several theorems showing relations between the three curves and involving questions of collinearity and concurrence. He then extends some of these results to the triangle and its circles and sets forth some sets of perspective triangles for which the axis of perspectivity passes through the center of perspectivity.

7. Professor H. S. White has determined all the triad sys-

tems in fifteen letters that have groups of substitutions into themselves, the number of types being 44.* Professors White and Cummings have also identified 33 groupless systems in fifteen letters, but a complete census of the groupless systems remained to be carried out. No method of doing this is known except that of actual sifting of all possible cases.

In forming triad systems in fifteen letters 1, 2, 3, \dots , 15 the seven triads that contain 1 may be written in the style 1 2 3, 1 4 5, 1 6 7, 1 8 9, 1 10 11, 1 12 13, 1 14 15. For the triads containing 2 there are then four types of opening

- (1) 2 4 6, 2 5 7, 2 8 10, 2 9 11, 2 12 14, 2 13 15;
- (2) 2 4 6, 2 5 7, 2 8 10, 2 11 12, 2 13 14, 2 9 15;
- (3) 2 4 6, 2 7 8, 2 5 9, 2 10 12, 2 13 14, 2 11 15;
- (4) 2 4 6, 2 7 8, 2 9 10, 2 11 12, 2 13 14, 2 5 15;

which may be designated as the triple tetrad, single tetrad, hexad, and duodecad types, respectively. Professor Cole has found that the only triad system in fifteen letters that can be made up from hexads and duodecads exclusively is the known system of Heffter. All others involve tetrads.

If to the tetrad 1 2 3, 1 4 5, 1 6 7; 2 4 6, 2 5 7 there be added the triads 3 4 7, 3 5 6, the result is a triad system in seven letters within that of the fifteen letters, which is then said to have a 7-head, or merely head. If 3 5 6 is replaced by 3 5 8, we may perhaps speak of a semi-head. The complete census of the triad systems in fifteen letters may then be tabulated as follows:

I. Types with triple tetrads, 60, of which 22 have heads, 12 have semi-heads but no heads, and 26 have neither head nor semi-head.

II. Types with tetrads but no triple tetrads, 19, of which 1 has a head, 3 have semi-heads but no heads, and 15 have neither head nor semi-head.

III. Heffter's headless system, 1.

Total, 80 systems.

The 23 types with heads all have groups; of the headless types 21 have groups and 36 are groupless.

8. Professor Glenn's first paper appeared in full in the December BULLETIN.

9. A paper by Professor Glenn published in the *Trans-*

* See this BULLETIN, vol. 21, pp. 164, 488.

actions in 1914 contains the determination of the expansion of the general binary form f of order $n(m+1) - 1$ as a binary m -ic in two arbitrary forms f_{1n}, f_{2n} of order n , the coefficients of the expansion being quantities φ_{in-1} ($i = 0, \dots, m$) of order $n - 1$. Assuming, in the present paper, that f_{1n}, f_{2n} are the universal covariants of a definite linear group G , it is shown that the φ_{in-1} are covariants; and a complete system of f under G is the simultaneous system of the set $\varphi_{in-1}, f_{1n}, f_{2n}$. Particular cases are treated; the most elementary being an explicit derivation of the (known) system of orthogonal concomitants of an m -ic.

As a result of applying methods similar to the above in the case when G is the group of binary linear transformations modulo p , $p \neq 2$, a superior limit for the order of an irreducible formal covariant modulo p of any binary form or set of forms is determined.

10. Professor Moore proposes to show that every two points of a continuous curve (no matter how crinkly it may be) can be joined by a *simple* continuous arc that lies wholly in the curve.

11. In his paper* "On the foundations of plane analysis situs," Professor R. L. Moore defines an open curve as a closed connected set of points M such that if P is any point of M , then $M - P$ is the sum of two mutually exclusive connected sets of points, neither of which contains a limit point of the other. He shows that if l is an open curve and S is the set of all points, then $S - l = S_1 + S_2$, where S_1 and S_2 are connected point sets, such that an arc from a point of S_1 to a point of S_2 contains at least one point of l . This theorem is proved on the basis of his set of axioms Σ_3 and is therefore true in certain spaces which are neither metrical, descriptive nor separable.

Dr. Kline proves the converse of this theorem for open curves. The statement of the converse theorem is as follows:

Suppose K is a closed set of points and that $S - K = S_1 + S_2$ where S_1 and S_2 are non-compact point sets such that

(1) every two points of S_i ($i = 1, 2$) can be joined by an arc lying entirely in S_i .

(2) every arc joining a point of S_1 to a point of S_2 contains a point of K .

* *Transactions*, vol. 17, no. 2, p. 131 (April, 1916).

(3) if O is a point of K and P is any point not belonging to K , then P can be joined to O by an arc having no point except O in common with K .

Every point set K that satisfies these conditions is an open curve.

12. If p is a prime of the form $4n + 3$, consider the number of quadratic residues included in the set $1, 2, \dots, 2n + 1$. In the present note Mr. Vandiver proves a theorem which sets forth a connection between this number and the number of quadratic residues in any set defined by $h + [ah] < p$, where a is a fixed integer less than $p - 1$ and h ranges over the set $1, 2, \dots, p - 1$, the expression $[ah]$ denoting the least positive residue of ah , modulo p . Analogous theorems are also found concerning the distribution of higher power residues.

13. Consider the indeterminate congruence of Lagrange

$$(x - 1)(x - 2) \cdots (x - (p - 1)) \equiv x^{p-1} - 1 \pmod{p},$$

where x is an indeterminate and p is a prime integer. Mr. Vandiver obtains some generalizations of this relation such that the set $1, 2, \dots, p - 1$ modulo p is replaced by all the incongruent residues of a composite ideal modulus which are prime to the modulus. The paper will appear in the *Annals of Mathematics*.

F. N. COLE,
Secretary.

CORRECTION.

THE following regrettable errata in the reports of the summer meeting and colloquium of the Society, published in the November BULLETIN, have been brought to the attention of the Secretary:

I. In the report of the summer meeting, page 65, it is stated that Professor C. N. Moore's paper appeared in full in the October BULLETIN. A paper with the same title did appear in the October BULLETIN, but it was read at the annual meeting held last January. The abstract of Professor Moore's summer meeting paper is printed below, with apologies to the author.