analogous to the greatest common divisor process. His results were published in the last issue of the Johns Hopkins Circular (July, 1915).

F. N. COLE,
Secretary.

WINTER MEETING OF THE SOCIETY AT COLUMBUS.

THE thirty-sixth regular meeting of the Chicago Section, being the fifth regular meeting of the American Mathematical Society in the west, was held at Columbus, Ohio, on Thursday, Friday and Saturday, December 30, 31, 1915, and January 1, 1916, in affiliation with the American Association for the Advancement of Science.

About one hundred persons were in attendance upon the various sessions, including the following sixty-seven members of the Society: Professor R. B. Allen, Professor Frederick Anderegg, Professor G. N. Armstrong, Professor R. P. Baker, Professor W. H. Bates, Professor P. P. Boyd, Professor Daniel Buchanan, Professor H. T. Burgess, Professor W. D. Cairns, Professor R. D. Carmichael, Professor H. E. Cobb, Professor Elizabeth B. Cowley, Dr. L. C. Cox, Professor D. R. Curtiss, Professor S. C. Davisson, Dr. W. W. Denton, Professor L. E. Dickson, Professor Peter Field, Professor B. F. Finkel, Professor T. M. Focke, Professor W. S. Franklin, Professor Harriet E. Glazier, Professor M. E. Graber, Professor Harris Hancock, Professor E. R. Hedrick, Dr. Cora B. Hennel, Dr. L. A. Hopkins, Professor L. C. Karpinski, Professor A. M. Kenyon, Mr. J. H. Kindle, Professor H. W. Kuhn, Professor Gertrude I. McCain, Dr. J. V. McKelvey, Dr. T. E. Mason, Professor F. E. Miller, Professor G. A. Miller, Professor J. A. Miller, Professor U. G. Mitchell, Professor C. N. Moore, Professor C. C. Morris, Professor F. R. Moulton, Professor A. D. Pitcher, Dr. V. C. Poor, Professor S. E. Rasor, Professor H. W. Reddick, Professor H. L. Rietz, Professor W. J. Risley, Professor W. H. Roever, Professor R. E. Root, Professor D. A. Rothrock, Professor F. H. Safford, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Dr. H. M. Sheffer, Professor H. E. Slaught, Professor K. D. Swartzel, Dr. E. H. Taylor, Mr. C. E. Van Orstrand, Professor C. A. Waldo, Professor C. J. West, Professor H. S. White, Professor E. J. Wilczynski, Professor

F. B. Wiley, Professor C. B. Williams, Professor B. F. Yanney, Professor A. E. Young, Professor Alexander Ziwet.

The opening session on Thursday afternoon was a joint meeting with Section A of the American Association, at which the program consisted of

(1) Retiring address of Professor H. S. White, Vice-President of Section A, on "Poncelet polygons."

(2) Retiring address of Professor E. J. Wilczynski, Chairman of the Chicago Section, "Some remarks on the historical development and the future prospects of the differential geometry of plane curves."

(3) Address by Professor W. W. Campbell, of the Lick Observatory, President of the American Association, on "The rotation of planetary nebulæ."

In the absence of Professor A. O. Leuschner, vice-president of Section A, Professor C. A. Waldo was called upon to preside at this meeting. At the other sessions of the Chicago Section, the chairman, Professor E. J. Wilczynski, presided, with occasional relief by Professor H. S. White, former president of the Society.

At the business meeting of the Chicago Section on Friday afternoon, the following officers of the Section were elected for the ensuing period of two years: Professor W. B. Ford, chairman; Professor Arnold Dresden, Secretary; Professor H. L. Rietz, third member of the program committee. In presenting the nominations for these offices, Professor G. A. Miller, chairman of the committee, announced that Professor H. E. Slaught, who had served as Secretary of the Section since 1907, had expressed a desire to be relieved at this time. On Professor Miller's motion, appreciation of Professor Slaught's services was indicated by a unanimous rising vote.

Professor R. D. Carmichael reported for the retiring program committee that no arrangements had been made for a symposium at the next meeting and recommended that this question be continued in the hands of the new program committee. This recommendation was adopted.

The joint dinner of Section A and the Chicago Section on Thursday evening was held at the Ohio Union, where the arrangements and service were unusually satisfactory and highly appreciated. About seventy were present at the dinner, which was followed by an evening of enjoyable social intercourse, interspersed with some informal remarks by

Professors H. S. White, E. R. Hedrick, H. E. Slaught, and others who were called on by the chairman.

On Friday morning, the Department of Mathematics of Ohio State University tendered a luncheon to all mathematicians in attendance at the meetings, thus affording another opportunity for a social gathering and placing all guests of the occasion under renewed obligation to their hosts.

By a rising vote at the final session Saturday afternoon the members of the Society acknowledged their deep obligation to the local committee under the direction of Professor S. E. Rasor, to the Department of Mathematics of Ohio State University, and to the University as a whole, for the very careful attention which had been given to all the provisions for these meetings and for the generous hospitality which had been extended in every way throughout the period.

The following papers were presented at this meeting:

(1) Professor H. T. BURGESS: "Note on the reduction of a family of quadratic forms."

(2) Professor J. B. SHAW: "Orthogonal vector systems in vector fields of three and more dimensions."

(3) Dr. A. J. KEMPNER: "On transcendental numbers."

(4) Dr. V. C. POOR: "A certain type of exact solutions of the equations of motion of a viscous liquid."

(5) Dr. V. C. POOR: "Transformation theorems in the theory of the linear vector function."

(6) Professor A. E. YOUNG: "On the determination of a certain class of surfaces."

(7) Professor C. J. WEST: "Note on nine-fold and four-fold correlation."

(8) Professor R. D. CARMICHAEL: "On a general class of series of the form $\Sigma\ C_n g(x + n)$."

(9) Dr. H. M. SHEFFER: "On a set of independent postulates for complex algebra."

(10) Dr. H. M. SHEFFER: "Mutually prime postulates."

(11) Professor C. H. SISAM: "On surfaces doubly generated by conics."

(12) Professor G. A. BLISS: "A note on the problem of Lagrange in the calculus of variations."

(13) Professor DANIEL BUCHANAN: "Three-dimensional periodic orbits of a particle subject to the attraction of a sphere having prescribed motion."

(14) Dr. W. V. LOVITT: "A type of singular points for a transformation of three variables."

(15) Dr. W. W. KÜSTERMANN: "Functions of bounded variation."

(16) Professor T. H. HILDEBRANDT: "Green's functions connected with general linear differential equations."

(17) Professor C. N. MOORE: "On the developments in Bessel's functions."

(18) Professor E. J. WILCZYNSKI: "Integral invariants in projective geometry."

(19) Professor G. A. MILLER: "Limits of transitivity of a substitution group."

(20) Professor G. A. MILLER: "Finite groups represented by special matrices."

(21) Professor R. P. BAKER: "The four-color map theorem."

(22) Mr. W. L. HART: "Differential equations and implicit functions in infinitely many variables."

(23) Professor S. E. RASOR: "On the integration of Volterra's derivatives."

(24) Mr. A. R. SCHWEITZER: "An apparent anticipation of Hilbert's conception of completeness."

(25) Mr. A. R. SCHWEITZER: "A bifurcative generalization of a functional equation due to Cauchy."

(26) Miss IDA M. SCHOTTENFELS: "A class of functions which are self-reciprocal in the sense of Mellin."

Mr. Hart was introduced by Professors E. H. Moore and F. R. Moulton. The papers of Professors Shaw, Sisam, Bliss, and Hildebrandt, Drs. Lovitt and Küstermann, Miss Schottenfels, and the first paper of Mr. Schweitzer, were read by title.

Abstracts of the papers follow in the order indicated in the above list of titles:

1. Let $\lambda A + B$ be the matrix of a family of quadratic forms in which A is non-singular. Subject the family successively to the two linear transformations whose matrices are H and T respectively. Then

$$H'(\lambda A + B)H = H'AHH^{-1}(\lambda I + A^{-1}B)H = R(\lambda I + N),$$

$$T'R(\lambda I + N)T = T'RTT^{-1}(\lambda I + N)T = M(\lambda I + N).$$

Professor Burgess points out that the matrix H may be so chosen that N is in the normal form and at the same time R is a matrix blocked off into principal minors each of which corresponds to an elementary divisor of $\lambda I + N$. The orders of the principal minors are the degrees of the corresponding elementary divisors; none of these principal minors overlap. He next shows that the matrix T may be so chosen that the normal form is unchanged and at the same time M differs from R in having all of its elements zero except those in the left principal diagonal of each of its blocks. As a consequence, all the well-known theorems on the elementary divisors of $\lambda A + B$ follow without further demonstration. The method of determining H and T is extremely simple and practical.

2. Professor Shaw's paper is a consideration of systems of three or more mutually orthogonal vectors in a field, such as the tangent, normal, and binormal of the vector lines, the normal and tangents of lines of curvature for a system of surfaces, and similar figures for polydimensional space. A linear vector operator of fundamental importance and much utility not hitherto noticed is exhibited, which for three-dimensional space is, in quaternion notation, α, β, γ being the moving orthogonal unit vectors,

$$\theta() = \rho() - (V\nabla\alpha)S\alpha - (V\nabla\beta)S\beta - (V\nabla\gamma)S\gamma,$$

where $2\rho = S\alpha\nabla\alpha + S\beta\nabla\beta + S\gamma\nabla\gamma$.

In terms of this operator the vector differential of any one of the three, in any one of the three directions is given by

$$(-S\lambda\nabla)\mu = V\mu\theta(\lambda).$$

The operator, of long-known use, $-S()\nabla \cdot \mu$, is given by

$$-S()\nabla \cdot \mu = V\mu\theta().$$

Application is made to congruences of lines.

3. Liouville (*Journal de Mathématiques*, volume 16, 1851) proved that $\sum_{r=0}^{r=\infty}\alpha_r/a^{m_r}$, where the α_r are real integers limited in absolute value and a a real integer $\geqq 2$, represents a transcendental number when the positive integral exponents $m_0, m_1, m_2, \cdots$ increase with sufficient rapidity. It is not

known, however. what constitutes this sufficient rapidity, so that the theorem does not serve to decide the transcendency of a number given in the form $\sum_{r=0}^{r=\infty}\alpha_r/a^{m_r}$ except in some extreme cases (including for example $m_r = r!, m_r = r^r$ (see Faber, *Mathematische Annalen*, volume 58, 1904).

Dr. Kempner proves that the power series

$$\sum_{r=0}^{\infty} \frac{\alpha_r}{a^{c^r}} \cdot x^r$$

has a transcendental value for every real rational value of x, if a and c are real integers $\geqq 2$, and if certain conditions are imposed on the integers α_r. These conditions are amply satisfied when: (1) $|\alpha_r| < M^r$, M arbitrary but fixed, and (2) only a finite number of the $\alpha_r = 0$.

4. The difficulty in obtaining exact solutions of the differential equations of a viscous liquid is due to their quadratic character. The most recent work of this kind has been done by C. W. Oseen. (See *Arkiv för Mathematik, Astronomi och Fysik*, volumes 3, 4, 6, 7, 9. Also *Acta Mathematica*, volume 34, 1911.) In the present paper, Dr. Poor proves the existence of a solution of the differential equations of a viscous liquid for all positive values of the time and in an infinite region. The body forces are assumed to be of the particular form

$$F(x, y, z, t) = F^{(1)}(x, y, z)e^{-k^2t}.$$

The method used is one of successive approximations. The successive steps involve solutions of sets of linear partial differential equations, the existence of which solutions is proved. Finally the convergence of the process is proved by using a dominance property of the successive steps. This dominance property persists if $F^{(1)}$ is properly restricted. Vector methods are used throughout the analysis.

5. Dr. Poor's second paper appeared in full in the January BULLETIN.

6. In two previous papers, Professor Young has discussed the problem of determining various classes of surfaces, taking as the starting point the fundamental equations written in the form first suggested and used by Bonnet in similar work. In

the present paper, he discusses from the same standpoint the general class of surfaces characterized by having $D = \pm D''$, using the customary notation, the lines of reference being lines of curvature.

7. Statistical data in the social sciences frequently cannot be classified into more than two or three broad classes. Professor West derives the working formulas for the application of the method of the correlation ratio to this type of correlation problems and discusses the value of the method as compared with certain other methods that have been proposed.

8. The series treated by Professor Carmichael are of the form

$$\Omega(x) = \sum_{n=0}^{\infty} c_n g(x+n), \qquad \bar{\Omega}(x) = \sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)},$$

where $g(x)$ is a function having the asymptotic character

$$g(x) \backsim x^{P(x)} e^{Q(x)} \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots \right)$$

valid for x approaching infinity in a positive sense along any line whatever parallel to the axis of reals, the functions $P(x)$ and $Q(x)$ being polynomials. For the special case in which $g(x) = 1/\Gamma(x)$ the series $(\bar{\Omega}x)$ is a factorial series. Professor Carmichael points out that the series $\Omega(x)$ and $\bar{\Omega}(x)$ are of great importance in investigating the properties of functions in the neighborhood of singularities of certain frequently occurring sorts and indicates his purpose to devote several memoirs to the development of a general theory of these series, especially in relation to the function-theoretic problem mentioned. In the present paper the foundation of a general theory of these series is laid. In case the series $\Omega(x)$ neither converges everywhere nor diverges everywhere its region of convergence [absolute convergence] is the half-plane $R(\sigma x) \leqq \lambda$ $[R(\sigma x) \leqq \mu]$, where $\lambda[\mu]$ is a constant depending on $c_0, c_1, c_2, \cdots$ and σ is the coefficient of the leading term in $Q(x)$ or in $P(x)$ according as $Q(x)$ is or is not of greater degree than $P(x)$. For the values of λ and μ explicit formulas are given analogous to the Cauchy-Hadamard formula for the radius of convergence of a power series. The paper contains also a treatment of uniform convergence, of the existence of singularities of the sum-function on the boundary of the region of convergence and of uniqueness of expansions in these series. A given function $f(x)$ has not more than one expansion $\Omega(x)$ or $\overline{\Omega}(x)$ when $g(x)$ is given and $R(\sigma) > 0$. For use in investigating this theory it was found convenient to employ a certain interesting generalization of the "generalized Dirichlet series"; and the theory of these generalized series was developed to the extent needed for the applications in question.

9. The postulate set for ordinary complex algebra presented by Dr. Sheffer is based on an undefined class and three undefined operations. The set differs from previous sets in that: (1) the number of undefined entities and of postulates is reduced; (2) the order relation is defined; and (3) the existence of 0, 1, negatives, reciprocals, and imaginaries is proved.

10. A set of m postulates such that no $m - 1$ of the postulates imply the mth is called independent. If P and Q are any two postulates of an independent set, P does not imply the whole of Q. P may imply, however, a part of Q. Two postulates, neither of which implies any part of the other, may be called mutually prime; and a set of postulates which are mutually prime by pairs may be called a set of mutually prime postulates. Obviously, mutual primeness implies independence; but not conversely. Also, mutual primeness implies E. H. Moore's complete independence; but not conversely. Dr. Sheffer shows how to construct sets of postulates which are mutually prime.

11. In this paper, Professor Sisam determines some fundamental properties of the surfaces of order seven and eight which contain two pencils of conics.

12. Professor Bliss's paper appeared in full in the February BULLETIN.

13. In an article entitled "A class of periodic orbits of an infinitesimal body subject to the attraction of n finite bodies" (*Transactions,* volume 8 (1907), pages 159–188), Longley discussed the periodic motion of a particle which moves subject to the Newtonian attraction of n finite bodies having pre-

scribed motion. The finite bodies and the particle are restricted to move in the one plane and the coordinates of the finite bodies, when referred to one of the bodies as origin, are assumed to be known functions of the time. In the present paper Professor Buchanan discusses the periodic motion of the particle when the finite bodies have the motion prescribed in Longley's article, but the particle is not restricted to move in a plane. Periodic solutions are determined as power series in a certain parameter which may be expressed as a function of the initial projection from the plane of motion of the finite bodies. These solutions have a period commensurable with the period of Longley's solutions and reduce to the latter when the initial projection from the plane of motion becomes zero.

14. Dr. Lovitt's paper appeared in full in the February BULLETIN.

15. The idea of a function of bounded variation, first developed for functions of one variable by Jordan, has been extended to two variables by Arzelà and Hardy in papers published in 1905. These authors' generalizations differ definitionally. Dr. Küstermann asks whether they coincide conceptually and shows that they do not by constructing a function which, while monotonically increasing in both x and y, and hence of bounded variation in Arzelà's sense, is not so according to Hardy's definition. Since both types of functions are integrable and can be developed into a double Fourier series it thus appears that Arzelà's generalization is not only the broader, but also the more natural one, preserving more closely the analogy to functions of a single variable. Nevertheless, in recent papers, Lebesgue and W. H. Young, apparently unacquainted with Arzelà's work, are using Hardy's definition. At any rate the designation "function of bounded variation in two variables" is not unique, but applies to-day to two distinctly different classes of functions.

16. The paper of Professor Hildebrandt is a generalization, in the sense of Moore's general analysis, of the memoir by Schlesinger: "Zur Theorie der linearen Integralgleichungen"*

* *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 24, pp. 84–123.

and that of Bounitzky: "Sur la fonction de Green des équations différentielles linéares ordinaires."* The differential equations treated are of the form:

$$(1)\qquad \frac{d}{dx}\,\eta(p'p''x) - \alpha(p'p''x) - J_{q''q'}\alpha(p'q''x)\eta(q'p''x) = 0,$$

$$(2)\qquad \frac{d}{dx}\,\eta(p'p''x) - J_{q''q'}\alpha(p'q''x)\eta(q'p''x) = 0,$$

where p' and p'' range over two general classes $\mathfrak{P}'$ and $\mathfrak{P}''$ respectively, x ranges over a bounded linear interval $x_0 \leqq x \leqq x_1$, α and η are suitably conditioned functions of these variables, and J is a linear operator on functions of the type $\kappa(q''q')$. The first part of the paper contains existence theorems, general solutions of the equations (1) and (2), a consideration of the analogue of the Fredholm determinant for the solutions, and a Green's theorem relating to the right-hand members of equations (1) and (2), and suitably defined adjoint expressions. The second part is devoted to the consideration of a system of Green's functions $\Gamma(p'p''x, y)$ which are continuous in x and y except for $x = y$, where the discontinuity is a function of the form $\kappa(p'p'')$. These Green's functions satisfy equations of the form (2), and certain general boundary conditions. The usual theorems relating to such functions are obtained, as well as the solutions of non-homogeneous equations of the form

$$\frac{d}{dx}\,\eta(p'p''x) = \alpha_0(p'p''x) + J_{q''q'}\alpha(p'q''x)\eta(q'p''x),$$

satisfying the given initial conditions.

17. There are certain methods of establishing the convergence of the developments in Bessel's functions that leave unsettled the question of the value to which they converge. One of these methods is the only means thus far available of establishing the convergence at the origin of the developments in Bessel's functions of order zero for the case where the function developed has discontinuities.† Hence, in order to have a complete treatment of the developments in question

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* *Journal de Mathématiques*, ser. 6, vol. 5 (1909), pp. 65–125.
† Cf. *Transactions*, vol. 12 (1911), p. 181.

that is adequate for the applications to mathematical physics and at the same time is not unnecessarily cumbersome, it is desirable to have a simple proof of the fact that the convergence is to the desired value. In Professor Moore's paper such a proof is given.

18. The simplest integral invariant of the projective theory of plane curves may be written in the form

$$H = \int \sqrt[3]{\theta_3}\, dx, \tag{1}$$

where

$$y''' + 3p_1y'' + 3p_2y' + p_3y = 0 \tag{2}$$

is the differential equation of the curve, and where

$$P_2 = p_2 - p_1^2 - p_1', \qquad P_3 = p_3 - 3p_1p_2 + 2p_1^3 - p_1'',$$
$$\theta_3 = P_3 - \tfrac{3}{2}P_2'.$$

Any other integral invariant of the curve may be expressed in the form $\int IdH$, where I is any one of its absolute differential invariants.

The integral H is as fundamental in the projective theory of plane curves as the length of arc, the fundamental integral invariant of the metric theory, is for metric geometry. The purpose of Professor Wilczynski's paper is to call attention to this fact and to provide a geometric interpretation for the integral H. His interpretation is as follows:

Consider any arc of the curve corresponding to the interval $a \leqq x \leqq b$ of the independent variable. Divide this interval into n parts by means of the values $x_0 = a$, x_1, x_2, $\cdots$ x_{n-1}, $x_n = b$ such that $\lim \delta x_k = \lim (x_{k+1} - x_k) = 0$ as n grows beyond bound. Let A, P_1, P_2, $\cdots$ P_{n-1}, B be the points on the curve which correspond to these $n + 1$ values of x. Let t_k be the tangent and C_k the eight-pointic nodal cubic of P_k. The three points of inflection of the cubic C_k are on a line i_k which intersects t_k in a point I_k. Denote by τ_k one of the inflectional tangents of C_k and let T_k be its intersection with t_k. The line P_kP_{k+1} will intersect i_k and t_k in two points I_k' and T_k', and the cross-ratio $(I_k', T_k', P_k, P_{k+1})$ turns out to be equal to

$$1 - \frac{3}{\sqrt[3]{10}} \sqrt[3]{\theta_3(x_k)}\, \delta x_k.$$

By a perspective correspondence the three points I'_{n-1}, T'_{n-1}, P_{n-1} of $P_{n-1}B$ may be projected into the points I'_{n-2}, T'_{n-2}, P_{n-1} of $P_{n-2}P_{n-1}$. Let B_{n-1} be the point of $P_{n-2}P_{n-1}$ which in this perspective corresponds to B. Then project similarly I'_{n-2}, T'_{n-2}, P_{n-2}, B_{n-1} into the four points I'_{n-3}, T'_{n-3}, P_{n-2}, B_{n-2} of $P_{n-3}P_{n-2}$, and continue in this way. We shall finally obtain upon the line AP_1 a point B_1 determined from B by this sequence of perspectives. As n grows beyond bound, B_1 will approach a limiting position Q on the initial tangent t_0 of the arc AB. Let κ denote the double-ratio $\kappa = (I_0, T_0, A, Q)$. We shall have

$$\log \kappa = \frac{3}{\sqrt[3]{10}} \int_a^b \sqrt{\theta_3}\, dx,$$

which gives the desired interpretation.

It is also easy to write down projective integral invariants in the theory of space curves and surfaces. None of these however have as yet received any interpretation.

19. On page 68 of this volume of the BULLETIN Professor Miller established the theorem that a substitution group of degree n which is neither alternating nor symmetric cannot be more than $3\sqrt{n} - 2$ times transitive when $n > 12$. In the present note he points out that it results from the main theorem proved in the article mentioned that such a group cannot be more than $\frac{5}{2}\sqrt{n} - 1$ times transitive. For large values of n this evidently gives a much smaller upper limit for the degree of transitivity than the one mentioned above, which is itself much smaller than the one commonly given, namely, $\frac{1}{3}n + 1$.

20. The direct object of Professor Miller's second paper is to prove that every finite group which contains an abelian subgroup of half its order can be represented by square matrices all of whose elements are equal to zero, with the exception of those which appear in one of the diagonals, and all of these are ordinary complex numbers which are different from zero. Incidentally several other somewhat general theorems are established. Among these are the following:

If G is an abelian group of order p^m and if H is a subgroup of order p^α, then a set of independent generators of G can always be so selected that at most $m - \alpha$ of them are not contained in H whenever G contains more than $m - \alpha$ independent generators. Moreover, it is always possible to construct a group having k independent generators and an arbitrary quotient group of order p^α such that the subgroup corresponding to identity of this quotient group cannot involve more than $k - \alpha$ of the operators in any possible set of independent generators of the group. If t transforms an abelian group G of order 2^m according to an automorphism of order 2 and if H is the subgroup of G formed by its operators which are invariant under t, then a set of independent generators of G can be so chosen that H contains all of them with the exception of at most three for each invariant of G/H.

21. The problem of the four-color map is taken by Professor Baker in the dual form. Every polyhedral net on the sphere can have its vertices marked with four colors so that no edge has the same color at its ends. The net is prepared and the problem reduced to that of an all-triangle polyhedron without triple circuits. This class is shown to be traversable by a closed curve passing once through all the vertices and having any pair of connected edges on the contour. For such a configuration a cardinal number relation is obtained for the number of successful colorings which is used as the basis of a two-step descending induction. The corresponding problem for one-sided closed surfaces of connectivity 2, 3, 4 is also solved. The numbers are 6, 6, 7.

22. At the April (1915) meeting of the Chicago Section, Mr. Hart presented a preliminary report which dealt with a part of the results of his present paper.

This investigation is concerned with functions f of the real variable $\xi = (x_1, x_2, \cdots)$ in the space

$$R: \quad M_1^{(i)} \leqq x_i \leqq M_2^{(i)} \qquad (M_1^{(i)}, M_2^{(i)} \leqq M; i = 1, 2, \cdots).$$

A function f is said to be completely continuous at the point ξ of R if, whenever

$$\lim_{n=\infty} x_{in} = x_i \qquad (i = 1, 2, \cdots),$$

it follows that

$$\lim f(\xi_n) = f(\xi) \qquad (\xi_n = x_{1n}, x_{2n}, \cdots).$$

The results obtained are of three sorts. In the first place, theorems on completely continuous functions are derived which include, for example, the Weierstrass theorem on uniformly convergent sequences of continuous functions and Taylor's theorem. In a second part of the paper the fundamental theorem of implicit function theory is proved for the infinite system of equations

$$(1) \qquad f_i(\xi, \eta) = 0 \quad [\xi \text{ in } R;\, \eta = (y_1, y_2, \cdots);\, \eta \text{ in } R],$$

which defines ξ as a function of $(y_1, y_2, \cdots)$. Then, finally, there is considered the infinite system of ordinary differential equations

$$(2) \qquad \frac{dx_i}{dt} = f_i(\xi, t)$$

$$(i = 1, 2, \cdots;\ x_i(t_0) = a_i;\ \xi \text{ in } R;\ |t - t_0| \leqq r_0),$$

and, under suitable hypotheses, the existence of a unique continuous solution $\xi(t)$ is established.

In the second and third parts of the paper the existence proofs are constructed by methods of successive approximation in which the theorems of the first section are of fundamental importance. The classical existence theorems for finite systems of implicit functions and differential equations are special cases of the results for systems (1) and (2).

23. In the *Rendiconti dei Lincei*, Volterra defined functions of lines, their continuity, and their derivatives. In later publications, he gave a method by means of Stokes' theorem for finding anti-derivatives for these functions of a line. The object of Professor Rasor's paper is to point out an instance quite analogous to the above from the calculus of variations, using Euler's equation for this purpose.

24. The object of Mr. Schweitzer's note is to call attention to Kempe's "law of continuity" which he phrases* as follows: "No entity is absent which can consistently be present." Kempe is careful to remark that the function of this law is to ensure the "complete definition" of his system (l. c., page 149) and that the law applies to "geometric sets" (page 177).

* *Proceedings London Math. Soc.*, vol. 21.

It appears* that Kempe's statement may desirably replace Hilbert's "axiom of completeness." Both principles emphasize the interdependence of mathematics and psychology and the problem presented by the relativity of the principles as used by their respective authors seems worthy of careful study.

25. Equivalent to Cauchy's well-known functional equation $\lambda(x+y) = \lambda(x) + \lambda(y)$ are the equations

$$(1) \qquad \lambda(x-z) - \lambda(x-y) = \lambda(y-z),$$

$$(2) \qquad \begin{aligned} \lambda(z) + \lambda(x+y) &= \lambda(x) + \lambda(y+z) \\ &= \lambda(y) + \lambda(x+z), \quad \lambda(0) = 0. \end{aligned}$$

In Mr. Schweitzer's second paper an interesting "bifurcation" is represented by the following generalizations:

$$(1') \quad \begin{aligned} f\{\lambda_1 f(t_1, t_2, \cdots, t_n, x_1), \cdots, \lambda_{n+1} f(t_1, t_2, \cdots, t_n, x_{n+1})\} \\ = \mu f(x_{i_1}, x_{i_2}, \cdots, x_{i_{n+1}}), \quad i_k = 1, 2, \cdots, (n+1), \end{aligned}$$

$$(2') \quad \begin{aligned} \mu_i \phi\{\lambda_1(x_1), \cdots, \lambda_j(x_{i-1}), \lambda_{j+1}(x_{i+1}), \cdots, \lambda_n(x_{n+1}), \\ \lambda_{n+1}\phi(x_i, t_1, t_2, \cdots, t_n)\} \\ = \phi\{\lambda_1(t_1), \lambda_2(t_2), \cdots, \lambda_n(t_n), \lambda_{n+1}\phi(x_1, x_2, \cdots, x_{n+1})\} \end{aligned}$$

where in the latter† system $i = 1, 2, 3, \cdots, (n+1)$, $x_0 \equiv x_2$, $x_{n+2} \equiv x_n$. Equations (1′) have been previously discussed by the author. In the case of equations (2′), the specialization $\lambda_1(x) = \lambda_2(x) = \cdots = \lambda_{n+1}(x) = \mu_i(x) = x$ leads to functional equations discussed by Abel, Stäckel, and Hayashi.

26. In the *Mathematische Annalen,* volume 68 (1910), pages 314–326, Mellin states the two following reciprocity theorems:

I. If $F(x)$ is any function belonging to a properly defined class and if

$$\varphi(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(x) t^{-x} dx,$$

* Cf. the "axiom of completeness" of G. Rabinovitch, BULLETIN, vol. 12 (1905–1906), p. 433 (abstract): "no motion is impossible unless it contradicts the above axioms."

† When $i = n + 1$, the argument of λ_n is x_n.

then

$$F(x) = \int_0^\infty \varphi(t) t^{x-1} dt.$$

II. If $\varphi(x)$ satisfies the conditions in Theorem I, and if $F(t) = \int_0^\infty \varphi(x) x^{t-1} dx$, then reciprocally,

$$\varphi(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(t) x^{-t} dt.$$

Example:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt,$$

$$e^{-t} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(x) t^{-x} dx \text{ for } a > 0; \; -\frac{\pi}{2} < \arg.\, t < \frac{\pi}{2}.$$

Miss Schottenfels' paper treats of a class of functions which are self-reciprocal in the above sense of reciprocity.

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ON PIERPONT'S DEFINITION OF INTEGRALS.

BY PROFESSOR M. FRÉCHET.

(Read before the American Mathematical Society, December 27, 1915.)

IN the second volume of his Lectures on the Theory of Functions of Real Variables, Professor J. Pierpont has given a new definition of Lebesgue integrals. This definition is interesting in as much as it realizes an effort to adapt the previous methods of presentation of Riemann integrals to the newer Lebesgue integrals.

But unfortunately the happiness of this idea is lessened in Pierpont's work by the choice of an inappropriate definition. Professor Pierpont intended to generalize the definition of Lebesgue integrals by defining upper and lower integrals of any function $f(x)$ on any linear set E_μ. Such definitions should not, of course, be arbitrary ones, and there are some primary conditions to be fulfilled, unless these definitions are to become quite artificial and uninteresting.