

Leçons sur la Théorie des Nombres. Par A. CHÂTELET. Paris, Gauthier-Villars, 1913. x+156 pp.

THIS little volume consists, as the preface tells us, of the Peccot Foundation lectures in form substantially as delivered at the Collège de France in the second semester of the year 1911-12. Following the line of development given in his thesis, "Sur certains ensembles de tableaux et leur application à la théorie des nombres" (*Annales scientifiques de l'École Normale Supérieure*, 1911), the author bases his exposition of the Dedekind theory of moduli and ideals largely upon the geometrical ideas of Minkowski and the so-called method of continued reduction of Hermite.

In the first chapter the algebraic foundation is laid by giving some necessary theorems concerning matrices and their relation to sets of forms, together with an all too brief account of Minkowski's theory of generalized distance. As in the thesis the term "*tableau*" is used throughout for a square matrix and the term matrix is employed to denote a rectangular array.

In the four following chapters the theory of Dedekind's moduli with its applications is studied in detail. If the sum and difference of two points be defined as in vector addition so that

$$\begin{aligned} (p_1, p_2, \dots, p_n) \pm (q_1, q_2, \dots, q_n) \\ = (p_1 \pm q_1, p_2 \pm q_2, \dots, p_n \pm q_n), \end{aligned}$$

the definition of a modulus of points in agreement with Dedekind's definition of a modulus of numbers follows naturally. It is then easy to show that the coordinates of the points of the simplest modulus of dimension m in a space of dimension n are given by the matrix equation

$$\| \|p_1, p_2, \dots, p_n\| = \| \|x_1, x_2, \dots, x_n\| \times A,$$

where the x 's are integers and A is a matrix with m rows and n columns. The modulus is said to be "type" if A exists with the elements of each row coordinates of a point of the modulus such that every point of the modulus is given by the matrix equation. The matrix A is called a "base" of the modulus. The criterion for a type modulus is that it has only a finite number of points all of whose coordinates are less in absolute value than a given number. In general there are only a finite

number of points whose distance from a given point of the modulus is finite. The points of the modulus form a lattice work.

Passing to realms of algebraic numbers, the author interprets the n conjugate values of a number $\bar{\omega}$ in a realm $K(\omega)$ as the coordinates of a point in space, real or semi-real, of n dimensions. The elementary theorem which affirms that $\bar{\omega}$ may be expressed as a linear integral function with rational coefficients of the n powers of a primitive element ω of $K(\omega)$, gives the matrix equation

$$||\bar{\omega}_1, \bar{\omega}_2, \dots, \bar{\omega}_n|| = ||a_0, a_1, \dots, a_{n-1}|| \times \Omega,$$

where the a 's are rational and Ω is the square root of the discriminant of ω . The numbers of the realm therefore give rise to a modulus R within which the integers of the realm form a type modulus C . The sub-modulus C is then according to Dedekind a multiple of R .

In the matrix equation for the conjugate values of an algebraic integer, the table Ω may be replaced by any other table

$$\Pi = R\Omega,$$

where the elements of R are rational and its determinant is not zero. It is then possible to determine π to a factor près in such a way as to establish an isomorphism between the numbers of the realm and the abelian system of tables

$$X_{\bar{\omega}} = \pi[\bar{\omega}_1, \bar{\omega}_2, \dots, \bar{\omega}_n]\pi^{-1}$$

with *rational* elements, where the bracket denotes a canonical table, i. e., a table in which the numbers $\bar{\omega}_i$ occupy the principal diagonal and all other elements are zero.

Similarly, when the integers of a realm are studied, it is easily shown that the integers of any ideal of the realm form a sub-modulus A of C which is again type and which is given by the matrix equation

$$||\beta_1, \beta_2, \dots, \beta_n|| = ||x_1, x_2, \dots, x_n|| \times PT,$$

where the x 's are integers, T is a base of the realm, and P is a table with rational elements. The table PT will be "a base relative to an ideal" if, and only if, the elements of the table

$$(PT)[\alpha_1, \alpha_2, \dots, \alpha_n](PT)^{-1}$$

are integers for every integer α of the realm.

In the two remaining chapters, devoted to continual reduction and the theorems of Minkowski and to the reduction of a base of a realm, geometrical ideas are brought even more clearly into evidence. In a modulus of points of dimension n in space of n dimensions, the points give rise to a totality of tables each of which is formed of the coordinates of n linearly independent points. Among the tables of such a totality it is natural that we should seek for the simplest. But what shall be the criterion? Geometrical intuition suggests at once that we choose those tables whose points lie as near as possible to the origin. But here a difficulty confronts us, for it may be that several points are equally distant from the origin and we cannot choose between them. These difficulties are overcome if for the ordinary distance we substitute Minkowski's "span," which is defined as the maximum of the absolute values of the differences between the coordinates of the two points. Thus the span $S(OA)$ of a point A from the origin is the maximum of the absolute values of its coordinates. Further, when two points have the same span with respect to the origin, we may distinguish between the ranks of their spans if we define "rank of span" to be the number reading from left to right of the maximum coordinate. For example, the two points A_1 (1, 5, 5) and A_2 (2, 3, 5) have the same span but the rank of A_1 is 2 while the rank of A_2 is 3.

A simplest or "reduced table" is one whose rows are the coordinates of n linearly independent points $A_i(p_1^{(i)}, p_2^{(i)}, \dots, p_n^{(i)})$ which satisfy the following conditions:

- (1) $S(OA_1) \leq S(OA_2) \leq \dots \leq S(OA_n),$
- (2) $p_1^{(i)} > 0,$
- (3) If $S(OA_i) = S(OA_{i+1}),$ rank $S(OA_i) \leq S(OA_{i+1}).$

For a given modulus the number of reduced tables is finite, as is geometrically evident. Nevertheless, it is possible to subject the modulus to a dilatation with continuous parameters so that for each of an infinite number of systems of values of the parameters a finite number of reduced tables exist. The importance of this method of introducing continuous parameters, which is Hermite's method of continuous reduction, lies in the fact that in a very important case, namely the case where the table is real and consequently associated with a decomposable form, the table is given only to a dilatation près.

When $n = 2$ and the elements of the table are real the series of reduced tables may be arranged, one for each interval, as a single parameter λ ranges both ways from a given value λ_1 defining a series of intervals such that

$$0 < \cdots < \lambda_{-2} < \lambda_{-1} < \lambda_1 < \lambda_2 < \cdots < \infty, \\ (\lim \lambda_i = \infty, \lim \lambda_{-i} = 0).$$

For the general case the reduction is accomplished by means of the fundamental inequalities of Minkowski. The two fundamental problems of finding the units of a realm and of separating the ideals of a realm into classes of equivalent ideals are made to depend upon the reduction of a base.

Three notes, the first on the application of the theory of moduli to periods of functions, the second a study of the realm $K(\sqrt{82})$, and the third a brief account of congruences with respect to an ideal and with respect to the norm of an ideal, occupy the last twenty-four pages of the book.

M. Châtelet modestly disclaims for the book any originality so far as material is concerned. But if it contains no hitherto unpublished results, the treatment is sufficiently novel to make the book a noteworthy contribution to the literature of algebraic numbers. The bringing together of the algebraic analysis of Hermite and the geometrical researches of Minkowski as aids to the development of the brilliant conceptions of Kummer and Dedekind is an achievement for which the mathematical world owes much to the author.

The book as a whole is well written, though at times it is brief almost to the point of obscurity. For the ordinary reader its value would have been greatly enhanced by additional concrete illustrations, and by even a few figures similar to those which illuminate Minkowski's *Diophantische Approximationen*.

E. B. SKINNER.

A Treatise on the Analytic Geometry of Three Dimensions. By GEORGE SALMON. Fifth edition, volume 2, edited by R. A. P. ROGERS. London, Longmans, Green and Company, 1915. xvi+334 pp.

THE second volume of the fifth edition of the *Treatise* begins with families of surfaces, which was Chapter XIII of the fourth edition. The numbering of the chapters has been retained.